

Bellman-Gronwall Inequality Approach to Oscillator Design for the Specific Bilinear Systems

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Abstract: In this paper, the existence of limit cycles for the specific bilinear systems is explored. Based on the Bellman-Gronwall inequality approach, not only the exponentially stable limit cycles phenomenon of such systems can be certified but also the oscillation behaviors of such systems can be correctly predicted. Finally, a numerical example is provided to illustrate the feasibility and effectiveness of the obtained result.

INTRODUCTION

Nonlinear system can offer oscillations with fixed amplitude and fixed frequency. These oscillations are named limit cycles, e.g., an RLC electrical circuit with a nonlinear resistor and Van der Pol equation. Limit cycles are singular phenomenon of nonlinear systems and have been a main interest of the researchers over the years; see, for example, [1-9], and the references therein. Prediction of limit cycles is very important, because limit cycles can appear in any kind of physical system. Ordinarily, a limit cycle can be desirable. This is the case of limit cycles in the electronic oscillators utilized in laboratories. There are at least two approaches to explore the phenomenon of limit cycles, namely describing function method and Poincare-Bendixson theorem. The disadvantages of the describing function approach are related to its approximate nature, and include the possibility of incorrect predictions. Besides, the Poincare-Bendixson theorem only offers a necessary condition to guarantee the existence of limit cycles. Consequently, if any one of the conditions of the Poincare-Bendixson theorem is not satisfied for some system, it can be guaranteed that there exists no limit cycles in such system. Conversely, even the conditions of the Poincare-Bendixson theorem are satisfied for some system, the existence of limit cycles cannot be guaranteed for such system [9].

In this paper, based on the Bellman-Gronwall inequality approach, the exponentially stable limit cycles for the specific bilinear control systems can be guaranteed. Furthermore, an estimate of the guaranteed convergence rate is also derived for such systems.

PROBLEM FORMULATION AND MAIN RESULT

In this paper, we consider the following bilinear control systems [10]:

$$\dot{z}(t) = Az(t) + ru(t)z(t), \forall t \geq t_0 \geq 0, \quad (1a)$$

$$z(t_0) = \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}, \quad (1b)$$

where

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathfrak{R}^2, \quad u \in \mathfrak{R},$$

$$A := \begin{bmatrix} -aw + d & -w(1 + a^2) \\ w & aw + d \end{bmatrix},$$

with $r \neq 0$, $a, d \in \mathfrak{R}$, $w \neq 0$, and $z(t_0) \neq 0$. Specially, the feedback control law is selected as follows

$$u(t) = -\frac{1}{r} \left[(z_1(t) + az_2(t))^2 + z_2^2 - k^2 + d \right], \quad (1c)$$

with $k > 0$. Thus the closed-loop systems are deduced as

$$\begin{cases} \dot{\mathfrak{z}}_1 = -awz_1 - w(1 - a^2)z_2 - z_1 \left[(z_1 + az_2)^2 + z_2^2 - k^2 \right], \\ \dot{\mathfrak{z}}_2 = wz_1 + awz_2 - z_2 \left[(z_1 + az_2)^2 + z_2^2 - k^2 \right], \quad \forall t \geq t_0. \end{cases}$$

$$x(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} z_1(t) + az_2(t) \\ z_2(t) \end{bmatrix}, \quad \forall t \geq t_0, \quad (2)$$

it can be obtained that

$$\begin{cases} \dot{\mathfrak{x}}_1(t) = -wx_2(t) - x_1(t) \left[x_1^2(t) + x_2^2(t) - k^2 \right], \\ \dot{\mathfrak{x}}_2(t) = wx_1(t) - x_2(t) \left[x_1^2(t) + x_2^2(t) - k^2 \right], \quad \forall t \geq t_0. \end{cases} \quad (3)$$

Obviously, $x = 0$ is an equivalent point of system (3), i.e., the solution of system (3) is given by $x(t) = 0$ if $x(t_0) = 0$ (or equivalently $z(t_0) = 0$). To avoid the trivial case of $x(t_0) = 0$, in the following, we only consider the system (1) under the case of $x(t_0) \neq 0$.

DEFINITION [9]

Consider the system (1). The closed and bounded manifold $\bar{s}(z) = 0$, in the $z_1 - z_2$ plane, is said to be an exponentially stable limit cycle if there exist two positive numbers α and β such that the manifold of $\bar{s}(z) = 0$ along the trajectories of system (1) satisfies the following inequality

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$$|\bar{s}(z(t))| \leq \beta \cdot \exp[-\alpha(t - t_0)] \forall t \geq t_0.$$

In this case, the positive number α is called the guaranteed convergence rate.

Now, we present the main result for the existence of limit cycles of system (1) as follows.

Theorem 1.

For the feedback bilinear systems (1), all of phase trajectories tend to the exponentially stable limit cycle $\bar{s}(z) = (z_1 + az_2)^2 + z_2^2 - k^2 = 0$ in the $z_1 - z_2$ plane, with the guaranteed convergence rate $2(z_{10} + az_{20})^2 + 2z_{20}^2$. Furthermore, the states $z_1(t)$ and $z_2(t)$ exponentially track, respectively, the trajectories

$$k \cos[w(t - t_0) + \theta_0] + ak \sin[w(t - t_0) + \theta_0]$$

and

$$k \sin[w(t - t_0) + \theta_0],$$

$$\theta_0 = \tan^{-1} \left(\frac{z_{20}}{z_{10} + az_{20}} \right),$$

in the time domain, with the guaranteed convergence rate $(z_{10} + az_{20})^2 + z_{20}^2$.

Proof. Define a smooth manifold $s(x) = 0$ and a continuous function

$$\theta(x) := \tan^{-1} \left[\frac{x_2}{x_1} \right]$$

with

$$s(x) := x^T x - k^2 = (z_1 + az_2)^2 + z_2^2 - k^2 = s(z).$$

Then the time derivatives of $s(x)$ and $\theta(x)$ along the trajectories of system (3) is given by

$$\begin{aligned} \frac{ds(x(t))}{dt} &= 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 \\ &= -2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - k^2); \end{aligned} \quad (4)$$

$$\frac{d\theta(x(t))}{dt} = \frac{\dot{x}_2 x_1 - x_2 \dot{x}_1}{x_1^2 + x_2^2} = w,$$

which imply

$$\theta(x(t)) = w(t - t_0) + \theta_0. \quad (5)$$

In the following, there are three cases to discuss the trajectories of the feedback control system of (3).

Case 1: $x_1^2(t_0) + x_2^2(t_0) = k^2$

In this case, by (4) and (5), it can be obtained that $\frac{ds(x(t))}{dt} = 0$, which implies

$$x_1^2(t) + x_2^2(t) = k^2, \forall t \geq t_0. \quad (6)$$

Hence we conclude that

$$x_1(t) = k \cos[w(t - t_0) + \theta_0], \quad \forall t \geq t_0;$$

$$x_2(t) = k \sin[w(t - t_0) + \theta_0], \quad \forall t \geq t_0;$$

$$s(x(t)) = 0, \quad \forall t \geq t_0,$$

in view of (5) and (6). Thus, from (2), it can be shown that

$$z_1(t) = k \cos[w(t - t_0) + \theta_0] + ak \sin[w(t - t_0) + \theta_0] \forall t \geq t_0;$$

$$z_2(t) = k \sin[w(t - t_0) + \theta_0], \quad \forall t \geq t_0;$$

$$\bar{s}(z(t)) = 0, \quad \forall t \geq t_0.$$

Case 2: $x_1^2(t_0) + x_2^2(t_0) > k^2$

In this case, by (4), it can be obtained that $s(x(t))$ is a strictly decreasing function of t with $s(x(t)) \geq 0, \forall t \geq t_0$, and

$$\frac{ds(x(t))}{dt} \leq -2k^2 s(x(t)), \forall t \geq t_0.$$

Applying the Bellman-Gronwall inequality with above differential inequality, one has

$$0 \leq s(x(t)) \leq s(x(t_0)) \cdot \exp[-2k^2(t - t_0)], \quad \forall t \geq t_0,$$

which implies

$$|s(x(t))| \leq s(x(t_0)) \cdot \exp[-2k^2(t - t_0)], \quad \forall t \geq t_0; \quad (7a)$$

$$\begin{aligned} k &\leq \sqrt{x_1^2(t) + x_2^2(t)} \\ &\leq \sqrt{k^2 + s(x(t_0)) \cdot \exp[-2k^2(t - t_0)]} \\ &\leq k + \sqrt{s(x(t_0)) \cdot \exp[-k^2(t - t_0)]}, \quad \forall t \geq t_0. \end{aligned} \quad (7b)$$

Consequently, by (5) and (7), we conclude that

$$\begin{aligned} |x_1(t) - k \cos[w(t - t_0) + \theta_0]| \\ \leq \sqrt{s(x(t_0)) \cdot \exp[-k^2(t - t_0)]}, \quad \forall t \geq t_0; \end{aligned}$$

$$\begin{aligned} |x_2(t) - k \sin[w(t - t_0) + \theta_0]| \\ \leq \sqrt{s(x(t_0)) \cdot \exp[-k^2(t - t_0)]}, \quad \forall t \geq t_0; \end{aligned}$$

$$|s(x(t))| \leq s(x(t_0)) \cdot \exp[-2k^2(t - t_0)], \quad \forall t \geq t_0.$$

Thus, from (2), it can be shown that

$$\begin{aligned} |z_1(t) - k \cos[w(t - t_0) + \theta_0] - ak \sin[w(t - t_0) + \theta_0]| \\ = |x_1(t) + ax_2(t) - k \cos[w(t - t_0) + \theta_0] \\ - ak \sin[w(t - t_0) + \theta_0]| \\ \leq |x_1(t) - k \cos[w(t - t_0) + \theta_0]| \\ + |a| \cdot |x_2(t) - k \sin[w(t - t_0) + \theta_0]| \\ \leq (1 + |a|) \cdot \sqrt{s(x(t_0)) \cdot \exp[-k^2(t - t_0)]}, \quad \forall t \geq t_0; \end{aligned}$$

$$\begin{aligned} |z_2(t) - k \sin[w(t - t_0) + \theta_0]| \\ \leq \sqrt{s(x(t_0)) \cdot \exp[-k^2(t - t_0)]}, \quad \forall t \geq t_0; \end{aligned}$$

$$|\bar{s}(z(t))| \leq s(x(t_0)) \cdot \exp[-2k^2(t-t_0)], \quad \forall t \geq t_0.$$

Case 3: $x_1^2(t_0) + x_2^2(t_0) = k_0^2 < k^2, k_0 \neq 0$

In this case, by (4), it can be obtained that $-s(x(t))$ is a strictly decreasing function of t with $-s(x(t)) \geq 0, \forall t \geq t_0$, and

$$\frac{d[-s(x(t))]}{dt} \leq -2k_0^2[-s(x(t))], \forall t \geq t_0.$$

Applying the Bellman-Gronwall inequality with above differential inequality, one has

$$0 \leq -s(x(t)) \leq -s(x(t_0)) \cdot \exp[-2k_0^2(t-t_0)], \quad \forall t \geq t_0,$$

which implies

$$\begin{aligned} k^2 + s(x(t_0)) \cdot \exp[-2k_0^2(t-t_0)] &\leq x_1^2(t) + x_2^2(t) \\ &\leq k^2, \forall t \geq t_0; \\ k - \sqrt{-s(x(t_0))} \cdot \exp[-k_0^2(t-t_0)] &\leq \sqrt{x_1^2(t) + x_2^2(t)} \\ &\leq k, \forall t \geq t_0. \end{aligned} \quad (8)$$

Consequently, by (5) and (8), we conclude that

$$\begin{aligned} &|x_1(t) - k \cos[w(t-t_0) + \theta_0]| \\ &\leq \sqrt{-s(x(t_0))} \cdot \exp[-k_0^2(t-t_0)]; \\ &|x_2(t) - k \sin[w(t-t_0) + \theta_0]| \\ &\leq \sqrt{-s(x(t_0))} \cdot \exp[-k_0^2(t-t_0)]. \end{aligned}$$

From (2), it results that

$$\begin{aligned} &|z_1(t) - k \cos[w(t-t_0) + \theta_0] - ak \sin[w(t-t_0) + \theta_0]| \\ &= |x_1(t) + ax_2(t) - k \cos[w(t-t_0) + \theta_0] \\ &\quad - ak \sin[w(t-t_0) + \theta_0]| \\ &\leq |x_1(t) - k \cos[w(t-t_0) + \theta_0]| \\ &\quad + |a| \cdot |x_2(t) - k \sin[w(t-t_0) + \theta_0]| \\ &\leq (1 + |a|) \cdot \sqrt{-s(x(t_0))} \cdot \exp[-k^2(t-t_0)], \quad \forall t \geq t_0; \\ &|z_2(t) - k \sin[w(t-t_0) + \theta_0]| \\ &\leq \sqrt{-s(x(t_0))} \cdot \exp[-k_0^2(t-t_0)]. \end{aligned}$$

This completes our proof.

Remark 1.

Obviously, by Theorem 1, the state of $x_1(t)$ can be represented as nonlinear oscillator with the amplitude $k\sqrt{1+a^2}$ and the frequency w rad/sec. The state of $x_2(t)$ can be also represented as nonlinear oscillator with the amplitude k and the frequency w rad/sec. Such oscillations are generally

independent of the initial condition and limit cycles of such oscillation are not influenced by parameter variation.

ILLUSTRATIVE EXAMPLE

Consider the bilinear control system of (1) with

$$a = 1, w = r = 2, d = 3, t_0 = 0,$$

and $z(0) = [4 \quad 4]^T$. By Theorem 1 with $k = 2$ and the control law

$$u(t) = -\frac{1}{2} [z_1^2(t) + 2z_2^2(t) + 2z_1(t)z_2(t) - 1],$$

we conclude that the exponentially stable limit cycle is given by $\bar{s}(z) = z_1^2 + 2x_2^2 + 2z_1z_2 - 4 = 0$ in the $z_1 - z_2$ plane, with the guaranteed convergence rate $\alpha = 160$. Furthermore, the states $z_1(t)$ and $z_2(t)$ exponentially track, respectively, the trajectories $2 \cos(2t + 0.46) + 2 \sin(2t + 0.46)$ and $2 \sin(2t + 0.46)$, in the time domain, with the guaranteed convergence rate $(z_{10} + az_{20})^2 + z_{20}^2 = 80$. Some state trajectories of the feedback-controlled system are depicted in Fig. (1) Fig. (3).

$z_1(t)$

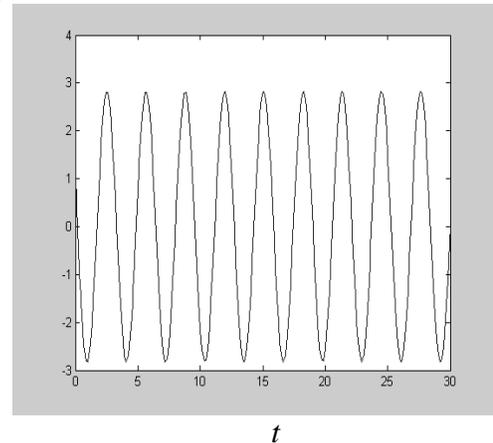


Fig. (1). $z_1(t)$ of the feedback-controlled system.

$z_2(t)$

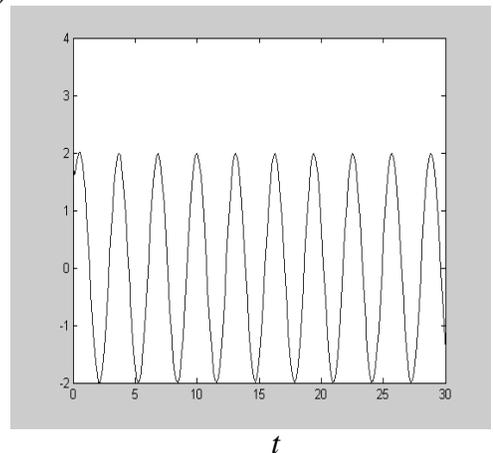


Fig. (2). $z_2(t)$ of the feedback-controlled system.

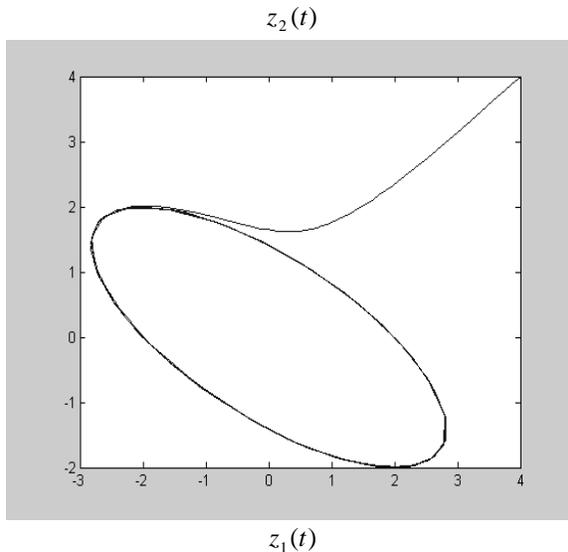


Fig. (3). $z_1(t)$ and $z_2(t)$ of the feedback-controlled.

CONCLUSIONS

In this paper, the existence of limit cycles for the specific bilinear control systems has been explored. Based on the Bellman-Gronwall inequality approach, not only the exponentially stable limit cycles phenomenon of such systems has been certified but also the oscillation behaviors of such systems has been correctly predicted. A numerical example has also been given to illustrate the feasibility and effectiveness of the obtained result. It is interesting to

of the obtained result. It is interesting to consider the oscillator design problem with more general systems.

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