Inverse Eigenvalue Problem for a Class of Spring-Mass Systems

Wenting Wan*

School of Mathematics and Physics, Jingchu University of Technology, Jingmen, Hubei, 448000, China

Abstract: This paper discusses the constructional problem for a class of spring-mass systems whose part particles are connected to the ground. The problem is converted to an inverse eigenvalue problem for Jacobi matrix. An inverse eigenvalue problem of determining the system from its some physical parameters and incomplete eigenpairs is solved. The necessary and sufficient condition for constructing the system uniquely with positive parameters is obtained. Furthermore, the concrete expressions of the solution and the related numerical algorithm are derived, and numerical results show that the algorithm is effective.

Keywords: Eigenpair, inverse eigenvalue problem, jacobi matrix, spring-mass system.

1. INTRODUCTION

Problems such as the longitudinal vibration of pile are often encountered in engineering practice. Along the long direction, rod is connected to the elastic foundation in such problems, and the rod vibration system can also be dispersed into spring-mass systems by using finite difference method. Inverse eigenvalue problems for spring-mass systems are the basic problems of dynamic inverse problems in mechanics, and related research has important application in vibration control, structural design, parameters identification and so on. The inverse eigenvalue problems for spring-mass systems are using part frequency (eigenvalues) and the corresponding vibration (eigenvectors) which have been measured to determine the remaining physical parameters of systems. Such problems in mathematics are converted to inverse eigenvalue problems for Jacobi matrices. At present, research on the inverse eigenvalue problems for Jacobi matrices had some good results in [1-6]. Gladwell and Gbadeyan [7], and Nylen and Uhlig [8] considered the inverse vibration problems of spring-mass systems only from two sets of eigenvalues. Using one eigenpair or two eigenpairs, the inverse eigenvalue problems for spring-mass systems were studied by Ram and Gladwell [9], and Tian and Dai [10]. Huang, *et al.* [11] studied by two incomplete eigenpairs to determine a spring-mass system, and the numerical method of the problem was given. In fact, it is difficult to obtain the entire spectrum of a system from vibration test in most situations, and only some of the eigenvalues and the eigenvectors of a system are got easily. Combined with the engineering of application, this paper constructs a grounding spring-mass system that is different from [10] by some physical parameters and incomplete eigenpairs of the system. Moreover, existence condition and expressions of the solution and the corresponding numerical algorithm are obtained.

N degree of freedom of spring-mass system whose anterior p particles are connected to the ground by springs (Fig. **1**). Generalized eigenvalue equation for the system is:

$$KX = \lambda MX,$$

where λ and X are respectively eigenvalue and eigenvector of matrix pair (K, M), $\lambda = \omega^2$, ω is natural frequency, X is vibration mode, mass matrix is:

$$M = diag(m_1, m_2, \cdots, m_n),$$

particle quality $m_i > 0$, ungrounded spring stiffness $k_i > 0 (i = 1, 2, \cdots, n-1)$, grounded spring stiffness $c_j > 0 (j = 1, 2, \cdots, p)$, stiffness matrix is:

$$K = \begin{bmatrix} c_1+k_1 & -k_1 & & & & & & \\ -k_1 & c_2+k_1+k_2 & -k_2 & & & & & \\ & & \ddots & & & & & \\ & & -k_{p-1} & c_p+k_{p-1}+k_p & -k_p & & & \\ & & & -k_p & k_p+k_{p+1} & -k_{p+1} & & \\ & & & & & \ddots & & \\ & & & & & -k_{n-2} & k_{n-2}+k_{n-1} & -k_{n-1} \\ & & & & & & -k_{n-1} & k_{n-1} \end{bmatrix}.$$

In this paper, we consider the following inverse eigenvalue problem:

Problem As for spring-mass system whose anterior p particles are connected to the ground by springs, particle quality $m_i > 0 (i = p+1, p+2, \cdots, n)$ and ungrounded spring stiffness $k_i > 0 (i = 1, 2, \cdots, n-1)$ are known. Given

$\lambda, \mu \in \boldsymbol{R}^+$ $(\lambda \neq \mu)$ and vectors

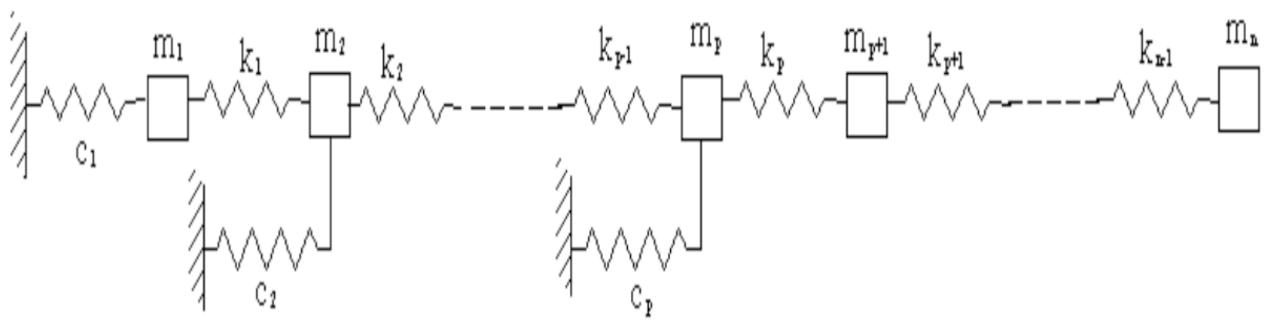

Fig. (1). A grounding spring-mass system.

$X_1=(x_1,x_2,\cdots,x_p)^T$,

$Y_1=(y_1,y_2,\cdots,y_p)^T\in\boldsymbol{R}^p$,

construct physical parameters of the system, that is, solve particle quality m_i, grounded spring stiffness

$c_i>0(i=1,2,\cdots,p)$ and vectors

$X_2=(x_{p+1},x_{p+2},\cdots,x_n)^T$,

$Y_2=(y_{p+1},y_{p+2},\cdots,y_n)^T\in\boldsymbol{R}^{n-p}$,

such that

$$KX=\lambda MX\ ,\ KY=\mu MY\ , \tag{1}$$

where vectors X and Y are

$X=(X_1^T,X_2^T)^T$, $Y=(Y_1^T,Y_2^T)^T\in\boldsymbol{R}^n$,

K and M are respectively mass matrix and stiffness matrix, (λ,X) and (μ,Y) are respectively the i'th and the j'th eigenpair of the system $(1\le i^{'},j^{'}\le n)$, when eigenvalues of the system are arranged in ascending order.

2. THE ANALYSIS OF PROBLEM

If $m_i>0(i=1,2,\cdots,n)$, then the equation $KX=\lambda MX$ is equivalent to the equation

$JX=\lambda X$,

where $J=M^{-1}K$ is $n\times n$ Jacobi matrix.

In [12], Gladwell gave the following definition and theorem.

Definition. For $n\times 1$ real vector

$X=(x_1,x_2,\cdots,x_n)^T$,

sign change number of sequence which composed of the component (that value being equal to zero can be neglected) is denoted by $S(X)$.

The eigenvalues of the Jacobi matrix are unequal real numbers, so let $\lambda_i>0(i=1,2,\cdots,n)$ be eigenvalues of $n\times n$ Jacobi matrix J .

If $\lambda_1<\lambda_2<\cdots<\lambda_n$, then

Lemma. Let (λ,X) be eigenpair of $n\times n$ Jacobi matrix J, then (λ,X) is the ith eigenpair of J if and only if

$S(X)=i-1$, $i=1,2,\cdots,n$.

Matrices K, M, X, Y can be expressed the following partitioned matrices.

$$K=\begin{bmatrix} K_1 & -k_pe_p\varepsilon_1^T \\ -k_p\varepsilon_1e_p^T & K_2 \end{bmatrix},$$

$$M=\begin{bmatrix} M_1 & \\ & M_2 \end{bmatrix},$$

$$X=\begin{bmatrix} X_1 \\ X_2 \end{bmatrix},\ Y=\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $p\times p$ Jacobi matrix K_1 is the top left corner matrix of K, $(n-p)\times(n-p)$ Jacobi matrix K_2 is the bottom right corner matrix of K, $p\times p$ diagonal matrix M_1 is the top left corner matrix of M, $(n-p)\times(n-p)$ diagonal matrix M_2 is the bottom right corner matrix of M, $p\times 1$ unit vector

$e_p=(0,0,\cdots,1)^T$,

$(n-p)\times 1$ unit vector

$\varepsilon_1 = (1,0,\cdots,0)^T$,

X_1, $Y_1 \in \boldsymbol{R}^p$, X_2, $Y_2 \in \boldsymbol{R}^{n-p}$.

formula (1) can be rewritten as follows.

$$\begin{cases} \lambda x_i m_i - x_i c_i = k_{i-1}(x_i - x_{i-1}) - k_i(x_{i+1} - x_i), \\ \qquad i = 1,2,\cdots,p, \\ (K_2 - \lambda M_2)X_2 = k_p x_p \varepsilon_1, \end{cases} \qquad (2)$$
$$\begin{cases} \mu y_i m_i - y_i c_i = k_{i-1}(y_i - y_{i-1}) - k_i(y_{i+1} - y_i), \\ \qquad i = 1,2,\cdots,p, \\ (K_2 - \mu M_2)Y_2 = k_p y_p \varepsilon_1. \end{cases}$$

where $k_0 = 0$, $x_0 = y_0 = 0$.

From (2), the problem solution can be transformed into solving the following problem I and Ⅱ:

Problem I. Given K_2, M_2 and positive real number λ, μ ($\lambda \neq \mu$), $X_1, Y_1 \in \boldsymbol{R}^p$ and k_p are known, solve vector X_2, $Y_2 \in \boldsymbol{R}^{n-p}$ which satisfied

$$\begin{cases} (K_2 - \lambda M_2)X_2 = k_p x_p \varepsilon_1, \\ (K_2 - \mu M_2)Y_2 = k_p y_p \varepsilon_1. \end{cases} \qquad (3)$$

Problem II. Given positive real number λ, μ ($\lambda \neq \mu$), X_1, $Y_1 \in \boldsymbol{R}^p$ and $k_i (i = 1,2,\cdots,p)$, solve m_i and $c_i (i = 1,2,\cdots,p)$, such that

$$\begin{cases} \lambda x_i m_i - x_i c_i = k_{i-1}(x_i - x_{i-1}) - k_i(x_{i+1} - x_i), \\ \mu y_i m_i - y_i c_i = k_{i-1}(y_i - y_{i-1}) - k_i(y_{i+1} - y_i), \end{cases} \qquad (4)$$

$i = 1,2,\cdots,p$,

where $k_0 = 0$, $x_0 = y_0 = 0$.

3. THE CONDITIONS OF UNIQUE SOLUTION OF THE PROBLEM

Let

$\varphi_{n-p}(\gamma) = \det(K_2 - \gamma M_2)$,

$\phi_0(\gamma) = 1$,

$\phi_1(\gamma) = k_{n-1} - \gamma m_n$,

$i \times i$ determinant $\phi_i(\gamma)$ is the top left corner cofactor of $\phi_{n-p}(\gamma)$, that is the $i \times i$ cofactor is obtained by deleting the first $n-p-i$ columns and the first $n-p-i$ rows of determinant $\det(K_2 - \gamma M_2)$.

Obviously, based on the result of $\varphi_{i-1}(\gamma)$, we can expand determinant $\phi_i(\gamma)$ by the first row and get:

$$\varphi_i(\gamma) = (k_{n-i} + k_{n-i+1} - \lambda m_{n-i+1})\varphi_{i-1}(\gamma) - k_{n-i+1}^2 \varphi_{i-2}(\gamma), \qquad (5)$$

$i = 2,3,\cdots,n-p$.

Let

$z_i = (\lambda - \mu) x_i y_i$

$f_i = x_i(y_{i+1} - y_i) - y_i(x_{i+1} - x_i)$,

$g_i = x_i(y_i - y_{i-1}) - y_i(x_i - x_{i-1})$,

$h_i = \lambda x_i(y_{i+1} - y_i) - \mu y_i(x_{i+1} - x_i)$,

$l_i = \lambda x_i(y_i - y_{i-1}) - \mu y_i(x_i - x_{i-1})$,

$r_i = k_i f_i - k_{i-1} g_i$,

$s_i = k_i h_i - k_{i-1} l_i$,

$i = 1,2,\cdots,p$.

Theorem 1. Systems of linear equations

$(K_2 - \gamma M_2)X_2 = k_p x_p \varepsilon_1$

have a unique solution if and only if

$\phi_{n-p}(\gamma) \neq 0$,

and the solution is:

$$x_i = x_p \phi_{n-p}^{-1}(\gamma) \phi_{n-i}(\gamma) \prod_{j=p}^{i-1} k_j,$$

$i = p+1, p+2, \cdots, n$.

Proof. Systems of linear equations

$(K_2 - \gamma M_2)X_2 = k_p x_p \varepsilon_1$

have a unique solution if and only if

$\det(K_2 - \gamma M_2) \neq 0$,

that is

$\phi_{n-p}(\gamma) \neq 0$.

If $\phi_{n-p}(\gamma)\neq 0$,

then,

$$X_2=(x_{p+1},x_{p+2},\cdots,x_n)^T$$
$$=k_p x_p(K_2-\gamma M_2)^{-1}\varepsilon_1$$
$$=k_p x_p[\det(K_2-\gamma M_2)]^{-1}\cdot(K_2-\gamma M_2)^*\varepsilon_1 .$$

Because $(K_2-\gamma M_2)^*\varepsilon_1$ is algebraic cofactor of the first row elements of the matrix $K_2-\gamma M_2$,

and then from (5), we have

$$X_2=(x_{p+1},x_{p+2},\cdots,x_n)^T$$
$$=k_p x_p\varphi_{n-p}^{-1}(\gamma)[\varphi_{n-p-1}(\gamma),\varphi_{n-p-2}(\gamma)\cdot k_{p+1},$$
$$\varphi_{n-p-3}(\gamma)\prod_{j=p+1}^{p+2}k_j,\cdots,\varphi_2(\gamma)\prod_{j=p+1}^{n-3}k_j,$$
$$\varphi_1(\gamma)\prod_{j=p+1}^{n-2}k_j,\varphi_0(\gamma)\prod_{j=p+1}^{n-1}k_j]^T\cdot$$

Therefore,

$$x_i=x_p\phi_{n-p}^{-1}(\gamma)\phi_{n-i}(\gamma)\prod_{j=p}^{i-1}k_j ,$$
$$i=p+1,p+2,\cdots,n .$$

From Theorem 1, the conditions for Problem I has unique solution is easy to derive.

Theorem 2. Problem I has a unique solution if and only if

$\phi_{n-p}(\lambda)\neq 0$,

and $\phi_{n-p}(\mu)\neq 0$,

and the solution is

$$x_i=x_p\varphi_{n-p}^{-1}(\lambda)\varphi_{n-i}(\lambda)\prod_{j=p}^{i-1}k_j,$$
$$y_i=y_p\varphi_{n-p}^{-1}(\mu)\varphi_{n-i}(\mu)\prod_{j=p}^{i-1}k_j, \tag{6}$$
$$i=p+1,p+2,\cdots,n .$$

Theorem 3. Problem II has a unique solution if and only if

$z_i\neq 0$,

and z_i, r_i, s_i have the same sign, the solution is

$$m_i=\frac{r_i}{z_i},\ c_i=\frac{s_i}{z_i}, \tag{7}$$
$$i=1,2,\cdots,p .$$

Proof. Systems of linear equations (4) have a unique solution if and only if coefficient determinant

$$\begin{vmatrix}\lambda x_i & -x_i\\ \mu y_i & -y_i\end{vmatrix}\neq 0 ,$$

that is $z_i\neq 0$,

and the solution is

$$m_i=\frac{r_i}{z_i},\ c_i=\frac{s_i}{z_i},$$
$$i=1,2,\cdots,p .$$

And for $m_i>0$, $c_i>0$,

z_i, r_i, s_i have the same sign, $i=1,2,\cdots,p$.

(λ,X) and (μ,Y) are respectively the i'th and the j'th eigenpair of the system $(1\le i',j'\le n)$, when eigenvalues of the system are arranged in ascending order. Combined with lemma, from Theorem 2 and Theorem 3, we have the necessary and sufficient condition for unique solution of the problem.

Theorem 4. Problem has a unique solution if and only if,

(1) $\phi_{n-p}(\lambda)\neq 0$ and $\phi_{n-p}(\mu)\neq 0$;

(2) $S(X)=i'-1$, $S(Y)=j'-1$;

(3) $z_i\neq 0$, and z_i, r_i, s_i have the same sign, $i=1,2,\cdots,p$.

4. NUMERICAL ALGORITHM AND EXAMPLE

The above discussion allows us to write down the algorithm to solve Problem, presented as follows.

Algorithm. Given diverse positive real number λ,μ (λ and μ are respectively the i'th and the j'th eigenvalue of the system $(1\le i',j'\le n)$, when eigenvalues of the system are arranged in ascending order.) , $m_i\in\boldsymbol{R}^+$ $(i=p+1,$ $p+2,\cdots,n)$, $k_i\in$ $\boldsymbol{R}^+$ $(i=1,2,\cdots,n-1)$, and $x_i,y_i(i=1,2,\cdots,p)$, construct $m_i,c_i\in\boldsymbol{R}^+$ $(i=1,2,\cdots,p)$ and vectors X_2, $Y_2\in\boldsymbol{R}^{n-p}$:

Step 1. From recurrence formula (5), compute $\phi_i(\lambda)$, $i=1,2,\cdots,n-p$. If $\phi_{n-p}(\lambda)=0$, go to step 7.

Step 2. From recurrence formula (5), compute $\phi_i(\mu)$, $i=1,2,\cdots,n-p$. If $\phi_{n-p}(\mu)=0$, go to step 7.

Step 3. From formula (6), compute x_i and y_i, $i=p+1,p+2,\cdots,n$.

Step 4. Compute $S(X)$, $S(Y)$. If $S(X)\neq i^{'}-1$ or $S(Y)\neq j^{'}-1$, go to step 7.

Step 5. Compute z_i, $i=1,2,\cdots,p$. If some $z_i=0$, $i=1,2,\cdots,p$, go to step 7.

Step 6. Compute r_i,s_i, $i=1,2,\cdots,p$. If z_i,r_i,s_i ($i=1,2,\cdots,p$) have different sign, go to step 7.

Step 7. The solution can not be determined uniquely, end the algorithm.

Step 8. From formula (7), compute m_i, c_i, $i=1,2,\cdots,p$,

Example. Given

$\lambda=2.2533, \mu=3.7120$, $n=8$, $p=3$,

$m_4=3$, $m_5=m_6=m_7=4$, $m_8=5$,

$k_1=1$, $k_2=k_3=2$, $k_4=3$,

$k_5=k_6=5$, $k_7=6$,

and $X_1=(x_1,x_2,x_3)^T = (-0.3336,0.5026,0.5421)^T$, $Y_1=(y_1,y_2,y_3)^T=(0.1852,-0.8192,0.4907)^T$.

We need to construct K, M, and design a 8 degree of freedom of the grounding spring-mass system. Moreover λ and μ are respectively the 5th and the 7th eigenvalue of the system.

By Algorithm, we get:

$\{\varphi_i(\lambda)\}_0^5=\{0.0010e+03,-0.0053e+03,-0.0465e$

$+03,0.0858e+03,1.0746e+03,-2.6636e+03\}$,

$\{\varphi_i(\mu)\}_0^5=\{0.0001e+04,-0.0013e+04,0.0012e$

$+04,0.0254e+04,-0.2049e+04,1.0286e+04\}$,

$\varphi_5(\lambda)\neq 0$ and $\varphi_5(\mu)\neq 0$,

$X_2=(x_4,x_5,x_6,x_7,x_8)^T$
$=(-0.4374,-0.1048,0.2837,0.1608,-0.1832)^T$,

$Y_2=(y_4,y_5,y_6,y_7,y_8)^T$

$=(-0.1955,0.0728,0.0176,-0.0899,0.0429)^T$.

The sign change number $S(X)=4$, $S(Y)=6$, and the results show that λ and μ is respectively the 5th and the 7th eigenvalue of the system.

$\{z_i\}_1^3=\{0.0901,0.6006,-0.3880\}$,

$\{f_i\}_1^3=\{0.1802,0.6907,0.1087\}$,

$\{g_i\}_1^3=\{0.0000,0.1802,0.6907\}$,

$\{h_i\}_1^3=\{0.1802,1.6036,0.9460\}$,

$\{l_i\}_1^3=\{0.0901,1.4053,1.5281\}$,

$\{r_i\}_1^3=\{0.1802,1.2012,-1.1641\}$,

$\{s_i\}_1^3=\{0.1802,1.8019,-1.1644\}$,

$z_i\neq 0$, and z_i, r_i, s_i have the same sign, $i=1,2,3$.

Then,

$\{m_i\}_1^3=\{2.0000,2.0000,3.0003\}$,

$\{c_i\}_1^3=\{1.9995,3.0002,3.0009\}$.

Therefore, $M=diag(2,2,3.0003,3,4,4,4,5)$,

$$K=\begin{bmatrix} 2.9995 & -1 & & & & & & \\ -1 & 6.0002 & -2 & & & & & \\ & -2 & 7.0009 & -2 & & & & \\ & & -2 & 5 & -3 & & & \\ & & & -3 & 8 & -5 & & \\ & & & & -5 & 10 & -5 & \\ & & & & & -5 & 11 & -6 \\ & & & & & & -6 & 6 \end{bmatrix}.$$

Using Matlab, we get all generalized eigenvalues of $KX=\lambda MX$ are

$\sigma(K,M)=\{0.0435,0.5969,1.2430,1.5271,$

$2.2533,3.0723,3.7120,4.5018\}$.

The eigenvector which corresponds to eigenvalue $\lambda=2.2533$ is,

$X=(-0.3335,0.5026,0.5421,-0.4375,$

$-0.1048,0.2837,0.1608,-0.1832)^T$

and the eigenvector which corresponds to eigenvalue $\mu=3.7120$ is:

$$Y=(0.1852,-0.8193,0.4907,-0.1955,$$

$$0.0728, 0.0176, -0.0899, 0.0429)^T .$$

These data indicate that the algorithm is very effective.

CONCLUSION

Some physical parameters of the grounding spring-mass system and two defective modes to construct real vibration system are presented in this paper. The problem comes down to the inverse eigenvalue problem for Jacobi matrix. The necessary and sufficient condition for the reconstruction of a physical vibration system with positive mass and stiffness elements from the known data is derived. And numerical algorithm and example are provided.

CONFLICT OF INTEREST

The author confirms that this article content has no conflict of interest.

ACKNOWLEDGEMENTS

This work is supported by the Key Scientific Research Project of Hubei Provincial Department of Education (No. D20134301), the Teaching Research Project of Jingchu University of Technology (No. JX2014-19)

REFERENCES

[1] G. M. L. Gladwell, "The inverse mode problem for lumped-mass systems", *Q. J. Mech. Appl. Math*., vol. 39, pp. 297-307, 1986.

[2] H. Dai, "Inverse eigenvalue problems for Jacobian and symmetric tridiagonal matrices", *Num. Math. J. Chinese Univ.*, vol. 12, no.1, pp. 1-13, 1990.

[3] X. Y. Hu, L. Zhang, and Z. Y. Peng, "On the construction of a Jacobi matix from its defective eigen-pair and a principal submatrix", *Math. Numer. Sin.*, vol. 22, no. 3, pp. 345-354, 2000.

[4] Z. Y. Peng, and X. L. Han, "Constructing Jacobi matrices with prescribed ordered defective eigenpairs and a principal submatrix", *J. Comput. Appl. Math*., vol. 175, pp. 321-333, 2005.

[5] X. Q. Wu, and E. X. Jiang, "A new algorithm on the inverse eigenvalue problem for double dimensional Jacobi matrices", *Linear Algebra Appl*., vol. 437, no. 7, pp. 1760-1770, 2012.

[6] Y. Wei, "A Jacobi matrix inverse eigenvalue problem with mixed data", *Linear Algebra Appl*., vol. 439, no. 10, pp. 2774-2783, 2013.

[7] G. M. L. Gladwell, and J. A. Gbadeyan, "On the inverse problem of the vibrating string or rod", *Q. J. Mech. Appl. Math.*, vol. 38, pp. 169-74, 1985.

[8] P. Nylen, and F. Uhlig, "Inverse eigenvalue problems associated with spring-mass systems", *Linear Algebra Appl.,* vol. 254, pp. 409-425, 1997.

[9] Y. M. Ram, and G. M. L. Gladwell, "Constructing a finite element model of a vibrating rod from eigendata", *J. Sound Vib.*, vol. 169, no. 2, pp. 229-237, 1994.

[10] X. Tian, and H. Dai, "An inverse eigenvalue problem for grounding spring-mass systems", *J. Shangdong Univ. (Nat. Sci.)*, vol. 43, no. 1, pp. 98-102, 2008.

[11] X. T. Huang, X. Y. Hu, and L. Zhang, "On the construct design of a fixed-fixed undamped mass-spring system under the sum of mass of system restriction", *Math. Appl.*, vol. 20, no.1, pp. 70-75, 2007.

[12] G.M.L. Gladwell, *Inverse Problems in Vibration*, Martinus Nijhoff Publishers: Dordrecht, the Netherland, 1986.

Received: November 28, 2014 Revised: January 10, 2015 Accepted: January 25, 2015