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Stabilization of Delay Systems with Nonlinearly Correlated Perturbations by a Constant Controller

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Abstract: In this paper we prove that the same design method of constant controllers for a delay systems given in the literature can be used to stabilize a minimum phase systems with nonlinearly correlated perturbations in the coefficients. This type of perturbation is induced by the composition with strictly positive real functions. This result can also be interpreted on the context of simultaneous and robust stabilization for single-input single-output, linear time invariant delay systems.

Keywords: Delay systems, robust stabilizability, simultaneous stabilizability, SPR functions, constant controllers.

1. INTRODUCTION

There are just a few results about simultaneous and robust stabilization of delay systems. For example, in [1] and [2] the authors establish delay-dependent results for robust stability and robust stabilization of uncertain linear systems with a time-varying state delay subject to norm-bounded uncertainty *via* memoryless state feedback. In [3-5], the authors present results of robust stability and robust stabilization, for uncertain linear systems with a time-varying state delay for which these results are independent of the size of the time-delay.

Some solutions to the problem of finding an algorithm to obtain a controller that simultaneously stabilizes m single-input single-output (SISO) linear time invariant (LTI) systems for the case of a constant controller using the state space description of the system have been given by [6] among others. However, the problem of simultaneous stabilization of delay systems remains without being studied. In [7] and [9], using a frequency domain approach it is consider the question of the existence of a rational controller, for the class of systems composed of a delay element e^{-hs} with (bounded and unbounded) uncertainty in h, followed by a plant characterized by a rational transfer function. There are given necessary and sufficient conditions for the existence of such controllers and a controller a design method is described. This controller design yields the entire set of all constant controllers which robustly stabilize a family of systems with uncertainty in the time-delay. Their results are independent of the size of the time-delay.

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Nevertheless, the problem of simultaneous and robust stabilization of delay systems with nonlinearly correlated perturbations in the coefficients remains an open problem. It is clear that this last problem is difficult and far from being solved. We will study the simple case when the controller is constant, and the delay systems belong to a particular class.

In this note we prove that the same design method of constant controllers given in [7], can be used with minimum-phase delay systems where their coefficients are nonlinearly correlated perturbations induced by the composition with strictly positive real (SPR) functions. Based on this fact, we present a result on simultaneous and robust stabilization for SISO LTI delay systems under nonlinearly correlated perturbations in their coefficients induced by the composition with SPR functions.

2. PRELIMINARIES

This section presents definitions and results which will be used throughout the paper.

Notation: Let $\mathbb{R}$ and $\mathbb{C}$ represent the field of real and complex numbers, respectively. Let $z \in \mathbb{C}$ such that $z = \sigma + j\omega$ with $\sigma \in \mathbb{R}$, $\omega \in \mathbb{R}$ and $j = \sqrt{-1}$. Also, let denote the real and imaginary part of the complex number z by $\mathrm{Re}[z] = \sigma$, $\mathrm{Im}[z] = \omega$. Consider the following sets

$$\begin{aligned}
\mathbb{C}^+ &\equiv \{z \in \mathbb{C} : \mathrm{Re}[z] > 0\} \\
\mathrm{Im}\,\mathbb{C} &\equiv \{z \in \mathbb{C} : \mathrm{Re}[z] = 0\}, \\
\overline{\mathbb{C}}_e^+ &\equiv \mathbb{C}^+ \cup \{\infty\} \cup \mathrm{Im}\,\mathbb{C} \\
\overline{\mathbb{C}}^+ &\equiv \mathbb{C}^+ \cup \mathrm{Im}\,\mathbb{C}, \\
\mathbb{C}_e^+ &\equiv \mathbb{C}^+ \cup \{\infty\}.
\end{aligned}$$

Definition 1 ([8]) A rational real function $p(s)$ of zero relative degreeis SPR (SPR0 function) if

i) $p(s)$ is analytic in $\mathrm{Re}[s] \geq 0$,

ii) $\mathrm{Re}[p(j\omega)] > 0$ for all $\omega \in \mathbb{R}$.

Now, we present the design method of constant controllers given in [7]. Let a delay system $\overline{p}(s)$ be represented as

$$\overline{p}(s) = p(s)e^{-hs} = \frac{N_p(s)}{D_p(s)}e^{-hs}$$

where $N_p(s)$ and $D_p(s)$ are polynomials as follows:

$$N_p(s) = n_r s^r + \cdots + n_0$$
$$D_p(s) = d_m s^m + \cdots + d_0$$

with $r \leq m$, we assume $N_p(s)$, $D_p(s)$ are coprime polynomials. Let a rational controller $C(s)$ be given by

$$C(s) = \frac{N_c(s)}{D_c(s)}$$

where $N_c(s)$, $D_c(s)$ are coprime polynomials. Furthermore, the delay system $\overline{p}(s)$ and controller $C(s)$ are connected by a standard unity feedback as shown in Fig. (**1**).

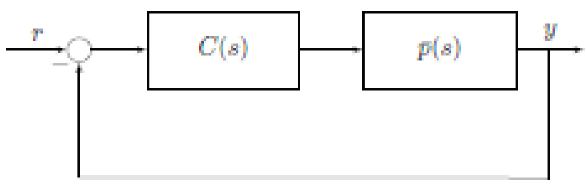

Fig. (1). Unity feedback. r: reference signal, y: system output, $C(s)$: controller, $\overline{p}(s)$: system.

The closed-loop system is given by

$$H(s) = \frac{N_c(s)N_p(s)e^{-hs}}{D_c(s)D_p(s) + N_c(s)N_p(s)e^{-hs}}.$$

and it is Hurwitz stable if

$$D_c(s)D_p(s) + N_c(s)N_p(s)e^{-hs} \neq 0$$

for $\mathrm{Re}[s] \geq 0$ for all $h \in [0, \infty)$.

A delay system is stabilized by a constant controller $C(s) = C_1$, if the following two conditions are satisfied:

C1) $D_p(s) + C_1 N_p(s) \neq 0$ in $\mathrm{Re}[s] \geq 0$.

C2) $|C_1| < \min_{\omega \in \mathbb{R}} \left| p^{-1}(j\omega) \right|$.

In this note we want to find a constant controller C_1 such that the closed-loop system

$$\mathrm{H}(s) = \frac{C_1 p(q(s))e^{-hs}}{1 + C_1 p(q(s))e^{-hs}} = \frac{C_1 \frac{N_p(q(s))}{D_p(q(s))}e^{-hs}}{1 + C_1 \frac{N_p(q(s))}{D_p(q(s))}e^{-hs}}$$

is stable for each SPR0 function $q(s)$. This problem is equivalent to to find a constant controller C_1 such that

$$D_q^{nm}(s)\left[D_p(q(s)) + C_1 N_p(q(s))e^{-hs}\right] \neq 0$$

for $\mathrm{Re}[s] \geq 0$ for all $h \in [0, \infty)$ for each SPR0 function $q(s)$ of order n, where $q(s) = \frac{N_q(s)}{D_q(s)}$.

3.THE MAIN RESULT

The main result is given in this section.

Theorem 2 Let $p(s) = \frac{N_p(s)}{D_p(s)}$ be a stable and minimum-phase rational function where $N_p(s)$, $D_p(s)$ are coprime polynomials. We assume the following conditions:

H1) $q(s) = \frac{N_q(s)}{D_q(s)}$ is any SPR0 function.

H2) The following inequalities are satisfied:

(a) $D_p(s) + C_1 N_p(s) \neq 0$ in $\mathrm{Re}[s] \geq 0$.

(b) $|C_1| < \min_{\omega \in \mathbb{R}} \left| p^{-1}(j\omega) \right|$.

Then:

i) There exists a constant controller C_1, such that, it robustly stabilizes $p(q(s))e^{-hs}$ for all $h \in [0, \infty)$, by continuously varying the coefficients of $q(s)$ in the set

$$G = \left\{ q(s) \;\middle|\; \begin{array}{l} q(s) = \dfrac{a_n s^n + \cdots + a_0}{b_n s^n + \cdots + b_0} \\ \text{is SPR0 function for} \\ (a_0, \ldots, a_n, b_0, \ldots, b_n) \in \mathrm{E}, \\ \mathrm{E} = Q_0 \times \cdots \times Q_n \times T_0 \times \cdots \times T_n \end{array} \right\}$$

where Q_i, T_i are closed intervals in $\mathbb{R}$ for $i = 0, \ldots, n$, i.e.,

$$D_p(q(s)) + C_1 N_p(q(s))e^{-hs} \neq 0$$

in $\mathrm{Re}[s] \geq 0$ for all $h \in [0, \infty)$ where $q(s) \in G$.

ii) C_1 simultaneously stabilizes $p(q_1(s))e^{-hs}, \ldots, p(q_m(s))e^{-hs}$ for all $h \in [0, \infty)$, if the orders of $q_1(s), \ldots, q_m(s)$ are different, where

$D_p(q_j(s)) + C_1 N_p(q_j(s))e^{-hs} \neq 0$

in $\text{Re}[s] \geq 0$ for all $h \in [0, \infty)$ and for $j = 1,...,m$ with $q_j(s) \in G$.

Proof. i) Observe that, since the denominator of the rational function $p(s)$ is Hurwitz stable, by Lemma 2 in [7], there exists a rational controller $C(s) = \dfrac{N_c(s)}{D_c(s)}$ such that

$D_c(s)D_p(s) + N_c(s)N_p(s)e^{-hs} \neq 0$

in $\text{Re}[s] \geq 0$ for all $h \in [0, \infty)$. Now, by Lemma 3 in [7], there also exists a constant robustly stabilizing controller C_0, such that

$$D_p(s) + C_0 N_p(s)e^{-hs} \neq 0$$

in $\text{Re}[s] \geq 0$ for all $h \in [0, \infty)$. Note that the constant controller C_0 is not unique. On the other hand, in [7], the authors establish that the constant controller C_1, stabilizes the delay system $p(s)e^{-hs}$ for all $h \in [0, \infty)$, i.e.,

$D_p(s) + C_1 N_p(s)e^{-hs} \neq 0$

in $\text{Re}[s] \geq 0$ for all $h \in [0, \infty)$, if conditions (C1) and (C2) are satisfied.

Then we need to verify that conditions (a) and (b) are satisfied by the rational function $p(q(s))$ for some constant controller C_1.

For condition (a): If the function $q(s)$ is a SPR0 function. Then the substitution of the variable s by the function $q(s)$ is an endomorphism over the Euclidian domain RH^∞, and these homomorphisms preserve sums, products, constants and Bezout's identities. Now from the fact that the SPR0 function $q(s)$ is analytic in $\text{Re}[s] \geq 0$ (see Definition 1) and by Theorem 9.9 in [10], we have that

$\text{Re}[q(s)] > 0 \quad \text{for} \quad \text{Re}[s] \geq 0.$

Then the condition (a), for the rational function $p(q(s))$ is satisfied, i.e.,

$D_p(q(s)) + C_1 N_p(q(s)) \neq 0 \quad \text{in} \quad \text{Re}[s] \geq 0.$

As a consequence, if $N_p(s) = n_r s^r + \cdots + n_0$, $D_p(s) = d_m s^m + \cdots + d_0$, $r \leq m$ and substituting $q(s)$ by $\dfrac{N_q(s)}{D_q(s)}$ in $D_p(s) + C_1 N_p(s) \neq 0 \text{inRe}[s] \geq 0.$

Since $N_q(s)$ and $D_q(s)$ are Hurwitz stable polynomials and by [11], then $D_q^{nm}(s)D_p(q(s)) \neq 0$ and $D_q^{nm}(s)N_p(q(s)) \neq 0$ in $\text{Re}[s] \geq 0$. Note also, that the homomorphism substitution preserves identities and also the zero element. Therefore the homomorphism substitution preserves the last inequality.

For condition (b): By Definition 2.1, condition 1, and since $q(s)$ is SPR0 function, then $q(\text{Im}\mathbb{C}) \subseteq \mathbb{C}^+$, $q(\text{Im}\mathbb{C}) \cap \text{Im}\mathbb{C} = \varnothing$ and $q(\text{Im}\mathbb{C})$ is a closed bounded domain. The function $p^{-1}(z)$ is analytic for any $z \in \mathbb{C}^+$ by our hypothesis about $p(s)$ being a minimum phase rational function. Since $p(s)$ is a stable and minimum-phase rational function, $p^{-1}(z) \neq 0$ for any $z \in \mathbb{C}^+$. Then, by the Minimum Module Theorem in [12], the minimum

$\min \left| p^{-1}(z) \right|$

for $z \in q(\text{Im}\mathbb{C})$ is reached for some $z \in q(\text{Im}\mathbb{C})$, since $q(\text{Im}\mathbb{C})$ is a closed bounded domain. Now, by the Minimum Module Theorem, the minimum

$\min \left| p^{-1}(z) \right|$

with $z \in \mathbb{C}^+ \cup \text{Im}\mathbb{C}$ is reached for some $z \in \text{Im}\mathbb{C}$. But, by Theorem 6 in [12],

$\left| p^{-1}(z) \right| > \min\limits_{z \in \mathbb{C}^+ \cup \text{Im}\mathbb{C}} \left| p^{-1}(z) \right|$

for every $z \in \mathbb{C}^+$. As a consequence, from the facts that $q(\text{Im}\mathbb{C}) \cap \text{Im}\mathbb{C} = \varnothing$, $q(\text{Im}\mathbb{C}) \subseteq \mathbb{C}^+$, Theorem 6 [12] and the Minimum Module Theorem

$$\min_{z \in \mathbb{C}^+ \cup \text{Im}\mathbb{C}} \left| p^{-1}(z) \right| = \min_{\omega \in \mathbb{R}} \left| p^{-1}(j\omega) \right| < \min_{z \in q(\text{Im}\mathbb{C})} \left| p^{-1}(z) \right|.$$

Using the condition (b), we have that

$\left| C_1 \right| < \min\limits_{\omega \in \mathbb{R}} \left| p^{-1}(j\omega) \right| < \min\limits_{z \in q(\text{Im}\mathbb{C})} \left| p^{-1}(z) \right|.$

Since

$\min\limits_{z \in q(\text{Im}\mathbb{C})} \left| p^{-1}(z) \right| \leq \min\limits_{\omega \in \mathbb{R}} \left| p^{-1}(q(j\omega)) \right|$

because $z \in q(\text{Im}\mathbb{C})$ is free, while $z \in \{z = q(j\omega) \mid \omega \in \mathbb{R}\}$ is restricted to the range of the complex function $q(j\omega)$ for each value $j\omega \in \text{Im}\mathbb{C}$. Then condition (b) is satisfied for the rational function $p(q(s))$ i.e.,

$\left| C_1 \right| < \min\limits_{\omega \in \mathbb{R}} \left| p^{-1}(q(j\omega)) \right|.$

Then, the two conditions (a) and (b) are satisfied. Now, since $q(s) \in G$, the constant controller C_1, robustly stabilizes the delay system $p(q(s))e^{-hs}$ for all $h \in [0, \infty)$.

ii) This proof is similar to the previous one using the same controller C_1, but now taking the orders of the SPR0 functions $q_1(s),...,q_m(s)$ different. ■

The assumption that the system is stable guarantees the existence of a constant controller by Lemma 2 and Lemma 3 in [7]. Note that this result is among the few results that exist about simultaneous stabilization and robust stabilization of delay systems using constant controllers with nonlinearly correlated perturbations in the coefficients induced by the composition of SPR0 functions. (see [11] and [13]).

Also, observe that conditions (a) and (b) are sufficient conditions for the constant controller C_1 to stabilizes the rational function $p(s)$. Therefore, the same sufficient conditions to stabilize the delay system $p(s)e^{-hs}$, with a constant controller C_1, are also sufficient to stabilize the delay system $p(q(s))e^{-hs}$, with the same constant controller C_1, for each SPR0 function $q(s)$ if the rational function $p(s)$ is minimum-phase rational function. Another consequence is that the design method presented in [7] for constant controllers which robustly stabilize the delay system $p(s)e^{-hs}$, it is also valid for the delay systems $p(q(s))e^{-hs}$ and can be extended to interval plants. Moreover, the unique additional condition for the rational function $p(s)$ is that it must be a minimum-phase rational function. On the other hand, Theorem **2** can be generalized to other contexts for SPR functions (see [14]).

4. EXAMPLE

In this section we give an examples to show the applicability of the result.

Example 1. In the example 1 of [7], the constant controllers which stabilizes

$$\bar{p}(s) = p(s)e^{-hs} = \frac{s+1}{s^2+5s+6}e^{-hs}$$

for all $h \in [0,\infty)$ are $C_1 \in (-4.56, 4.56)$. Based on this example, we show that these C_1, simultaneously stabilize

$$\bar{p}_1(s) = p_1(s)e^{-hs} = \frac{10s^2+17s+6}{63s^2+119s+63}e^{-hs}$$

and

$$\bar{p}_2(s) = p_2(s)e^{-hs} = \frac{5(2s^4+21s^3+68s^2+69s+20)}{63s^4+653s^3+2127s^2+2255s+750}e^{-hs}$$

for all $h \in [0,\infty)$, also. Note that $p_1(s) = p\left(\frac{3s+5}{2s+1}\right)$ and $p_2(s) = p\left(\left(\frac{s+3}{s+5}\right)\left(\frac{3s+5}{2s+1}\right)\right)$. Also note that $\frac{2s+3}{5s+7}$ is an SPR0 function, and $\left(\frac{s+3}{s+5}\right)\left(\frac{3s+5}{2s+1}\right)$ is an SPR0 function. Then, by Theorem **2**, the constant controllers $C_1 \in (-4.56, 4.56)$ simultaneously stabilize $\bar{p}_1(s)$ and $\bar{p}_2(s)$ for all $h \in [0,\infty)$. On the other hand, we can verify that

$$|C_1| = 4.56 \le \min_{\omega} \left|p_1^{-1}(j\omega)\right| = 6.34$$

and

$$|C_1| = 4.56 \le \min_{\omega} \left|p_2^{-1}(j\omega)\right| = 6.11$$

based on the example 1 in [7] too.

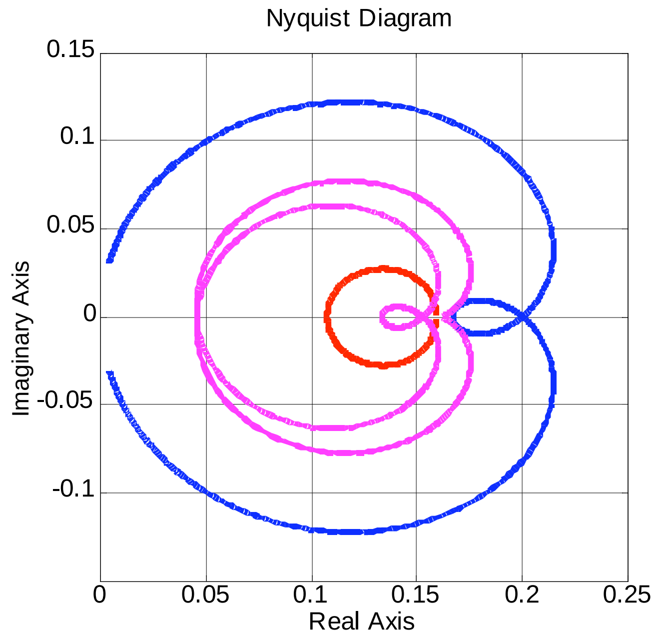

Fig. (2). Nyquist diagram for the Example 1 systems. \$p(j\omega)\$: solid blue line, \$p_1(j\omega)\$: dashed red line, \$p_2(j\omega)\$: dotted magenta line.

As an observation, the controller design condition, implies that the largest gain needed to cross the point (1,0) on the Nyquist diagram for $|p^{-1}(j\omega)|$, notice that for p_1 and p_2, the composition of p with their respective SPR functions, their Nyquist diagrams are contained inside the Nyquist diagram of p, as can it be seen in Figure 2, from here it is possible to visualize why the gain for the controller will stabilize the plants in the example.

In Fig. (**3**) we see the Nyquist diagrams for the delayed systems for which the stability is ensure by the constant controller C_1.

CONCLUSIONS

Based on some results of a delay systems class, we prove that the same design method of constant controllers given in the literature stabilizes delay systems with nonlinearly correlated perturbations in the coefficients induced by the composition with SPR functions, if the plant is minimum-phase. Based on this fact, we have given a result on simultaneous and robust stabilization for SISO LTI delay systems, with nonlinearly correlated perturbations in the coefficients, induced by the composition with SPR0 functions, using constant controllers. Moreover, the result works for almost any SPR function. Also is possible to find the entire set of all constant controllers which robustly stabilize of the system, for each SPR0 function, using the technics found in the literature.

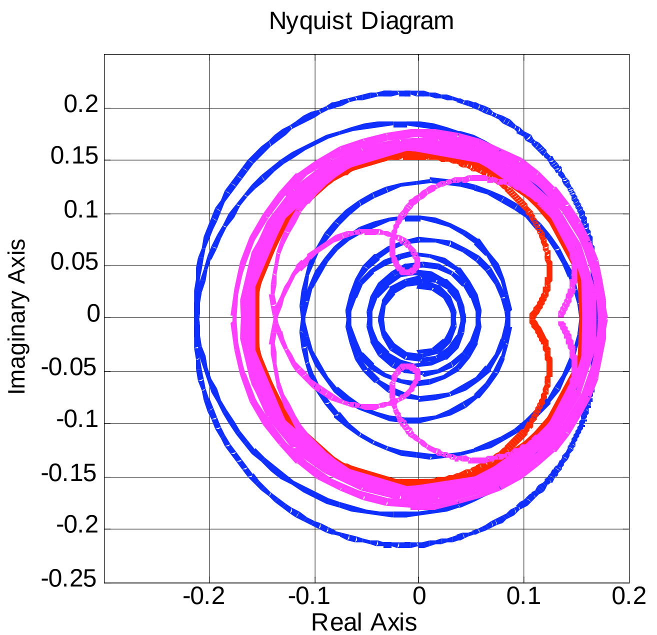

Fig. (3). Nyquist diagram for the Example 1 systems. For $h = 1$, $\overline{p}(j\omega)$: solid blue line, $\overline{p}_1(j\omega)$: dashed red line, $\overline{p}_2(j\omega)$: dotted magenta line.

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