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Echo in a Semi-Bounded Plasma

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Abstract: Plasma echo theory is revisited and reviewed to apply it to a semi-bounded plasma. Spatial echoes in a semi-bounded plasma are investigated by calculating the electric field produced by external charges and satisfying the boundary conditions at the interface. We show that echoes can occur at various spots. The diversity of echo occurrence spots is due to the boundary terms.

Keywords: Boundary condition, Echo, Semi-bounded plasma.

I. INTRODUCTION

Plasma echoes in an infinite plasma have long been known theoretically [1, 2] as well as experimentally [3]. Let us briefly review the echo theory. The distribution function of plasma particles $f(k,\omega,v)$ in Fourier space has a singularity at $\omega = kv$ (in addition to other singularities). In its inversion to (x,t) variables, this singularity modulates the distribution function with the exponential phase $e^{ik(x-vt)}$ or $e^{-i\omega(t-x/v)}$. This modulation is called free streaming term since $x = vt$ is the characteristic line of a free particle. This term makes the modulation of the distribution function more and more oscillatory as t or x increases, and consequently, $\int fdv$ will become vanishingly small due to almost complete cancelations (phase-mixing). Therefore, the free streaming term yields no appreciable effect on macroscopic variable such as density perturbation.

However, if two free streaming terms are multiplied as in a nonlinear determination of the second order electric field, it is evident from the expression $e^{ik_1(x_1-vt_1)}e^{ik_2(x_2-vt_2)}$ that a constructive interference can result in at a certain time (temporal echo) (or at a certain spot; spatial echo) such that $k_1x_1 + k_2x_2 = v(k_1t_1 + k_2t_2)$. In this case, the exponential phase of the second order distribution function vanishes and the corresponding velocity integral does not phase-mix, resulting in an echo.

In this work, we investigate spatial echoes in semi-bounded plasma to find the locations where the echoes can be detected. In a bounded plasma, the electric field is determined by satisfying the boundary conditions at the interface. We find that a bounded plasma gives diversity of echo locations as compared to an infinite plasma. It appears that the diversity is due to reflections of waves at the boundary.

The content of this work is organized as follows. In Section II, the nonlinear Vlasov and Poisson equations are expanded in perturbation series to prepare iteration with the linear solutions. In Section III, linear solutions are obtained by carefully observing the kinetic and the electric boundary conditions at the interface. In Section IV, the second order quantities are analyzed to find the echo locations. Lastly, Section V touches upon a mathematical aspect of the differential equation and its Fourier transform.

2. FORMULATION OF THE PROBLEM

We consider a plasma consisting of electrons and stationary ions, the latter forming the uniform background. The plasma is assumed to occupy the half-space $x \geq 0$. The region $x < 0$ is assumed to be a vacuum. The perturbed electron distribution function $f(x,v,t)$ and the electric field $E(x,t)$ will depend on x-coordinate only since y and z coordinates have translational invariance. We have the nonlinear Vlasov equation and the Poisson equation to describe the electrostatic perturbation:

$$\frac{\partial}{\partial t} f(v,x,t) + v\frac{\partial f}{\partial x} - \frac{e}{m}E(x,t)\frac{\partial f}{\partial v} = 0 \tag{1}$$

$$\frac{\partial E}{\partial x} = 4\pi\left(-e\int dvf + \rho_0(x,t)\right) \tag{2}$$

where ρ_0 represents the external charges:

$$\rho_0(x,t) = \rho_1 e^{i\omega_1 t}\delta[k_0(x-L_1)] + \rho_2 e^{-i\omega_2 t}\,\delta[k_0(x-L_2)] \tag{3}$$

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k_0 is introduced to make the argument of the ! -- function dimensionless. We solve the simultaneous Eqs. (1) and (2) for a given $\rho_0(x,t)$ as prescribed by Eq. (3). In mathematical terms, we have an inhomogeneous system, driven by the source term in Eq. (3). The responses f and E should be determined by ρ_0.

The kinetic equation is supplemented by the kinematic boundary condition which we assume to be the specular reflection condition:

$$f(v,0) = f(-v,0) \tag{4}$$

Assuming that the external perturbation is small, we solve Eqs. (1) and (2) by successive approximation. First, the linear solution of Eq. (1) will be obtained for f with the boundary condition Eq. (4). Substituting this solution in Eq. (2) yields an integral equation for the electric field which is solved by Fourier transform. Then the linear solution will be used to obtain the higher order solutions. We work only up to the second order. The higher order distribution function should also satisfy the boundary condition Eq. (4). The electric field should satisfy the electric boundary condition: The electric displacement $D(x)$ is continuous across the interface.

Let us Fourier transform the above equations with respect to t to write:

$$\begin{aligned}&-i\omega f(v,x,\omega)+v\frac{\partial f}{\partial x}-\frac{e}{m}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\\&E(x,\omega-\omega')\frac{\partial}{\partial v}f(x,v,\omega')=0 \quad (x>0)\end{aligned} \tag{5}$$

$$\frac{\partial}{\partial x}E(x,\omega)=4\pi\left[-e\int_{-\infty}^{\infty}f(x,v,\omega)dv+\rho_0(x,\omega)\right](x>0) \tag{6}$$

$$\begin{aligned}&\rho_0(x,\omega)=2\pi[\rho_1\delta(\omega+\omega_1)\delta(k_0(x-L_1))+\\&\rho_2\delta(\omega-\omega_2)\,\delta(k_0(x-L_2))]\end{aligned} \tag{7}$$

In this work, the Fourier transform is defined by:

$$f(k,\omega)=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dtf(x,t)e^{-ikx+i\omega t}$$

$$f(x,t)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}f(k,\omega)e^{ikx-i\omega t}$$

Note that Eqs. (5) and (6) are valid only in the region $x>0$, the plasma region. Eqs. (5) and (6) constitute a set of nonlinear simultaneous equations . We solve the set of equations by successive approximations in terms of perturbation series:

$$f(x,v,\omega)=f_0(v)+f^{(1)}(x,v,\omega)+f^{(2)}(x,v,\omega)+\cdots$$

$$E(x,\omega)=E^{(1)}(x,\omega)+E^{(2)}(x,\omega)+\cdots$$

Breaking down Eqs. (5) and (6) order by order, we have

$$-i\omega f^{(1)}(x,v,\omega)+v\frac{\partial f^{(1)}}{\partial x}=\frac{e}{m}E^{(1)}(x,\omega)\frac{df_0}{dv} \quad (x>0) \tag{8}$$

$$\frac{\partial}{\partial x}E^{(1)}(x,\omega)=4\pi\left[-e\int_{-\infty}^{\infty}f^{(1)}(x,v,\omega)dv+\rho_0(x,\omega)\right](x>0) \tag{9}$$

$$\begin{aligned}&-i\omega f^{(2)}(x,v,\omega)+v\frac{\partial f^{(2)}}{\partial x}=\frac{e}{m}E^{(2)}(x,\omega)\frac{df_0}{dv}+\\&\frac{e}{m}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}E^{(1)}(x,\omega-\omega')\frac{\partial}{\partial v}f^{(1)}(x,v,\omega')\end{aligned} \tag{10}$$

$$\frac{\partial}{\partial x}E^{(2)}(x,\omega)=-4\pi e\int_{-\infty}^{\infty}f^{(2)}(x,v,\omega)dv \quad (x>0) \tag{11}$$

The above equations should be solved in accordance with the following boundary conditions.

Specular reflection boundary condition:

$$\begin{aligned}&f^{(1)}(x=0,v,\omega)=f^{(1)}(x=0,-v,\omega),\\&f^{(2)}(x=0,v,\omega)=f^{(2)}(x=0,-v,\omega)\end{aligned} \tag{12}$$

Electric field boundary condition: Electric displacement $D(x)$ is continuous across the interface $x=0$,

$$D^{(1)}(0^+)=D^{(1)}(0^-)=E_0, \quad D^{(2)}(0^+)=D^{(2)}(0^-) \tag{13}$$

where $D^{(1)}(0^-)$ equals to the vacuum electric field E_0.

3. LINEAR SOLUTION

Eq. (8) was solved as a first order linear differential equation by earlier authors [4, 5]. (See section V.) The equation also can be dealt with by direct Fourier transform after a suitable extension of $E(x)$ because $E(x)$ in Eq. (8) is defined only for the region $x>0$. Eq. (8) is invariant with respect to the simultaneous reflections $x\longrightarrow -x$ and $v\longrightarrow -v$ provided $E(x)$ is an odd function, i.e., $E(x)$ is continued into region $x<0$ in an odd function manner ($f_0(v)$ is an even function of v): $E(-x)=-E(x)$. Then, the specular reflection boundary condition is automatically satisfied because we have $f(x,v)=f(-x,-v)$. Furthermore, Eqs. (8) and (9), extended into the negative x region in this way, can be solved by Fourier transform (with respect to the variable x) as usual. From Eq. (8), we obtain:

$$f^{(1)}(k,\omega,v)=\frac{ie}{m}\frac{1}{\omega-kv}\frac{df_0}{dv}E^{(1)}(k,\omega) \tag{14}$$

Multiplying Eq. (9) by e^{-ikx} and integrating piecewisely from $-\infty$ to 0^- and 0^+ to ∞, one obtains:

$$ikE^{(1)}(k,\omega)+N^{(1)}=-4\pi e\int_{-\infty}^{\infty}f^{(1)}(k,\omega,v)dv+4\pi\rho_0(k,\omega) \tag{15}$$

$$\textit{where} \quad N^{(1)}=E^{(1)}(0^-,\omega)-E^{(1)}(0^+,\omega)$$

$$\rho_0(k,\omega)=\frac{2\pi}{k_0}[\,\rho_1e^{-ikL_1}\delta(\omega+\omega_1)+\rho_2e^{-ikL_2}\delta(\omega-\omega_2)] \tag{16}$$

Note that the discontinuity of $E(x)$ at $x=0$ is accounted in the above Fourier transformed equation (15). Using Eq. (14) in Eq. (15) yields:

$$E^{(1)}(k,\omega)=\frac{i}{k\,\varepsilon(k,\omega)}[N^{(1)}-4\pi\rho_0(k,\omega)] \tag{17}$$

$$\varepsilon(k,\omega)=1+\frac{\omega_p^2}{k}\int_{-\infty}^{\infty}dv\,\frac{\frac{df_0}{dv}}{\omega-kv} \tag{18}$$

which is the dielectric function (ω_p is the plasma frequency). $N^{(1)}$ is determined from the electric boundary condition as shown in the sequel.

We need the electric displacement $D(x)$ to enforce the boundary condition Eq. (13). By definition, $D(k,\omega)=E(k,\omega)+\frac{4\pi i}{\omega}J(k,\omega)$ where J is the current, $J(k,\omega)=-e\int_{-\infty}^{\infty}dv\,v\;f(k,\omega,v)$. Using Eq. (14), we obtain

$$D^{(1)}(k,\omega)=\varepsilon(k,\omega)\,E^{(1)}(k,\omega)=\frac{i}{k}[N^{(1)}-4\pi\rho_0(k,\omega)] \tag{19}$$

Inverting Eq. (19) gives:

$$D^{(1)}(x,\omega)=-\frac{1}{2}N^{(1)}S(x)+\frac{4\pi^2}{k_0}[\rho_1\delta(\omega+\omega_1)S(x-L_1)+\rho_2\delta(\omega-\omega_2)S(x-L_2)]\quad(x>0) \tag{20}$$

where we used the formula [6]

$$\int_{-\infty}^{\infty}dk\,\frac{e^{ikx}}{k}=i\pi S(x)$$

where $S(x)$ is the step function: $S(x)=1$ if $x>0$ or -1 if $x<0$. Therefore, we obtain:

$$D^{(1)}(0^+,\omega)=-\frac{1}{2}N^{(1)}-\frac{4\pi^2}{k_0}[\rho_1\delta(\omega+\omega_1)+\rho_2\delta(\omega-\omega_2)] \tag{21}$$

Setting this quantity to E_0, we have:

$$N^{(1)}=-2E_0-\frac{8\pi^2}{k_0}[\rho_1\delta(\omega+\omega_1)+\rho_2\delta(\omega-\omega_2)] \tag{22}$$

Using Eq. (22) in Eq. (17) gives:

$$E^{(1)}(k,\omega)=\frac{-2i}{k\,\varepsilon(k,\omega)}\left[\begin{array}{l}E_0+\frac{4\pi^2}{k_0}(\rho_1\delta(\omega+\omega_1)(1+e^{-ikL_1})\\+\rho_2\delta(\omega-\omega_2)(1+e^{-ikL_2}))\end{array}\right] \tag{23}$$

In Eq. (23), we determined $E^{(1)}$ in terms of the vacuum field and the external charges. E_0 and '1' in the factors $(1+e^{-ikL_1})$ and $(1+e^{-ikL_2})$ come from the boundary condition which will be absent from an infinite plasma.

In an infinite plasma without boundary, we would obtain:

$$E^{(1)}(k,\omega)=\frac{-4\pi i\rho_0(k,\omega)}{k\,\varepsilon(k,\omega)}=\frac{-i}{k\varepsilon(k,\omega)}\frac{8\pi^2}{k_0}(\rho_1\delta(\omega+\omega_1)e^{-ikL_1}+\rho_2\delta(\omega-\omega_2)e^{-ikL_2})$$

4. THE SECOND ORDER SOLUTION AND ECHO OCCURRENCE

Next, we deal with the second order equations, Eqs. (10) and (11). After Fourier transforming with respect to x, we obtain:

$$f^{(2)}(k,\omega,v)=\frac{i}{\omega-kv}\frac{e}{m}(\frac{\partial}{\partial v}Q(k,\omega,v)+\frac{df_0}{dv}E^{(2)}(k,\omega)) \tag{24}$$

where Q stands for:

$$Q(k,\omega,v)=\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}E^{(1)}(\omega-\omega',k-k')\,f^{(1)}(\omega',k',v) \tag{25}$$

Fourier transforming Eq. (11) and and using Eq. (24) therein give:

$$E^{(2)}(k,\omega)=\frac{1}{k\varepsilon(k,\omega)}\left[-\frac{4\pi e^2}{m}\int_{-\infty}^{\infty}\frac{dv}{\omega-kv}\frac{\partial Q}{\partial v}+i\,N^{(2)}\right] \tag{26}$$

where $N^{(2)}$ is the jump of $E^{(2)}(x)$ at $x=0$. We determine it from the boundary condition in Eq. (13). We have

$$D^{(2)}(\omega,k)=E^{(2)}(\omega,k)+\frac{4\pi i}{\omega}J^{(2)}(\omega,k)$$

$$J^{(2)}(\omega,k)=-e\int dv\,vf^{(2)}(\omega,k,v) \tag{27}$$

Using Eq. (24) for J, we obtain:

$$D^{(2)}(\omega,k)=\varepsilon E^{(2)}(\omega,k)+\frac{4\pi e^2}{\omega m}\int dv\,\frac{v}{\omega-kv}\frac{\partial Q}{\partial v} \tag{28}$$

Using Eq. (26) in the above equation yields:

$$D^{(2)}(\omega,k)=\frac{i}{k}N^{(2)}-\frac{4\pi e^2}{mk\omega}\int dv\,\frac{\partial Q}{\partial v} \tag{29}$$

Clearly the last integral vanishes. The boundary condition Eq. (13) requires $N^{(2)}$ to be zero, and we write, after integration by parts:

$$E^{(2)}(k,\omega)=\frac{4\pi e^2}{m\varepsilon(k,\omega)}\int_{-\infty}^{\infty}dv\,\frac{Q(k,\omega,v)}{(\omega-kv)^2} \tag{30}$$

Let us summarize the equations that will be used to investigate the echo occurrence condition:

$$E^{(2)}(k,\omega)=\frac{ie^3}{m^2}\frac{1}{\varepsilon(k,\omega)}\int_{-\infty}^{\infty}dv\,\frac{\frac{df_0}{dv}}{(\omega-kv)^2}\int_{-\infty}^{\infty}d\omega'\int_{-\infty}^{\infty}dk'\,\frac{E^{(1)}(\omega',k')E^{(1)}(\omega-\omega',k-k')}{\omega'-k'v} \tag{31}$$

$$E^{(1)}(k,\omega)=\frac{-2i}{k\varepsilon(k,\omega)}\left[\begin{array}{l}E_0+\frac{4\pi^2}{k_0}(\rho_1(b+e^{-ikL_1})\\ \delta(\omega+\omega_1)+\rho_2(b+e^{-ikL_2})\\ \delta(\omega-\omega_2))\\ \end{array}\right] \tag{32}$$

where the letter $'b'$ stands for unity; it will be eventually put to 1 but indicates that it is a boundary term. We write the product terms which survive the phase-mixing if a suitable condition is met:

$$\frac{E^{(1)}(\omega',k')E^{(1)}(\omega-\omega',k-k')}{\omega'-k'v}=$$

$$\frac{-\alpha}{k'(k-k')(\omega'-k'v)\varepsilon(k',\omega')\varepsilon(k-k',\omega-\omega')}\times$$

$$\rho_1\rho_2\delta(\omega'-\omega_2)\delta(\omega-\omega'+\omega_1)e^{-ik'L_2}e^{-i(k-k')L_1} \tag{33}$$

$$\rho_1\rho_2\delta(\omega'+\omega_1)\delta(\omega-\omega'-\omega_2)e^{-ik'L_1}e^{-i(k-k')L_2} \tag{34}$$

$$b\,\rho_1^2\delta(\omega'+\omega_1)\delta(\omega-\omega'+\omega_1)e^{-i(k-k')L_1} \tag{35}$$

$$b\,\rho_2^2\delta(\omega'-\omega_2)\delta(\omega-\omega'-\omega_2)e^{-i(k-k')L_2} \tag{36}$$

$$b\,\rho_1\rho_2\delta(\omega'-\omega_2)\delta(\omega-\omega'+\omega_1)e^{-i(k-k')L_1} \tag{37}$$

$$b\,\rho_1\rho_2\delta(\omega'+\omega_1)\delta(\omega-\omega'-\omega_2)e^{-i(k-k')L_2} \tag{38}$$

where $\alpha=(8\pi^2/k_0)^2$. Terms (33) to (38) are to be added. The terms multiplied by b are boundary terms which are irrelevant to an infinite plasma.

We show first that two terms (33) and (34) can yield an echo resonance. Adding the two terms, we have:

$$\frac{E^{(1)}(\omega',k')E^{(1)}(\omega-\omega',k-k')}{\omega'-k'v}=-\alpha\rho_1\rho_2\delta(\omega+\omega_1-\omega_2)\times$$

$$\frac{\delta(\omega'+\omega_1)e^{-ik'L_1}\,e^{-i(k-k')L_2}+\delta(\omega'-\omega_2)e^{-ik'L_2}\,e^{-i(k-k')L_1}}{k'(k-k')(\omega'-k'v)\varepsilon(k',\omega')\varepsilon(k-k',\omega-\omega')} \tag{39}$$

Integral $\int d\omega'$ in Eq. (31) can be done easily owing to the δ --functions:

$$E^{(2)}(k,\omega)=-i\,A\int_{-\infty}^{\infty}\frac{dv}{(\omega-kv)^2}\frac{df_0}{dv}\frac{\delta(\omega+\omega_1-\omega_2)}{\varepsilon(k,\omega)}$$

$$\int_{-\infty}^{\infty}\frac{dk'}{k'(k-k')}\times$$

$$\left[\frac{e^{-ik'L_2}\,e^{-i(k-k')L_1}}{(\omega_2-k'v)\varepsilon(k',\omega_2)\,\varepsilon(k-k',-\omega_1)}-\frac{e^{-ik'L_1}\,e^{-i(k-k')L_2}}{(\omega_1+k'v)\varepsilon\,(k',-\omega_1)\varepsilon(k-k',\omega_2)}\right] \tag{40}$$

$$A=\alpha\,\frac{e}{m}\frac{\omega_p^2}{4\pi}\rho_1\rho_2 \tag{41}$$

In $\int dk'$ --integral, poles at $k'=0$ and $k'=k$ do not produce the echo resonance terms; poles at the roots of the dielectric functions contribute negligibly to the integral, long after the Landau damping time; poles at $k'=\omega_2/v$ and $k'=-\omega_1/v$ produce the resonance terms. Therefore, picking up the residues at $k'=\omega_2/v$ and $k'=-\omega_1/v$, $\int dk'$ --integral can be done:

$$E^{(2)}(k,\omega)=2\pi A\int_{-\infty}^{\infty}\frac{dv}{(\omega-kv)^2}\frac{df_0}{dv}\frac{\delta(\omega+\omega_1-\omega_2)}{\varepsilon(k,\omega)}\times$$

$$\left[-\frac{e^{-iL_2\frac{\omega_2}{v}}\,e^{-i(k-\frac{\omega_2}{v})L_1}}{\omega_2(k-\frac{\omega_2}{v})\varepsilon(\frac{\omega_2}{v},\omega_2)\varepsilon(k-\frac{\omega_2}{v},-\omega_1)}+\frac{e^{i\frac{\omega_1}{v}L_1}\,e^{-i(k+\frac{\omega_1}{v})L_2}}{\omega_1(k+\frac{\omega_1}{v})\varepsilon(-\frac{\omega_1}{v},-\omega_1)\varepsilon(k+\frac{\omega_1}{v},\omega_2)}\right] \tag{42}$$

$E^{(2)}(x,t)$ can be obtained by inverting the above equation. $\int d\omega$ -integral can be done immediately, and we write:

$$E^{(2)}(x,t)=\frac{A}{2\pi}e^{-i\omega_3 t}\int_{-\infty}^{\infty}dv\,\frac{df_0}{dv}\int_{-\infty}^{\infty}dk\,\frac{e^{ikx}}{\varepsilon(k,\omega_3)(\omega_3-kv)^2}\times$$

$$\left[-\frac{e^{-iL_2\frac{\omega_2}{v}}\,e^{-i(k-\frac{\omega_2}{v})L_1}}{\omega_2(k-\frac{\omega_2}{v})\varepsilon(\frac{\omega_2}{v},\omega_2)\varepsilon(k-\frac{\omega_2}{v},-\omega_1)}+\frac{e^{i\frac{\omega_1}{v}L_1}\,e^{-i(k+\frac{\omega_1}{v})L_2}}{\omega_1(k+\frac{\omega_1}{v})\varepsilon(-\frac{\omega_1}{v},-\omega_1)\varepsilon(k+\frac{\omega_1}{v},\omega_2)}\right] \tag{43}$$

where $\omega_3=\omega_2-\omega_1$. In the above $\int dk$ --integral, the non-phase-mixing contribution comes from the double pole at $k=\omega_3/v$. The residue is obtained by differentiating the integrand with respect to k and putting $k=\omega_3/v$. Here we differentiate only the exponential factor since it gives the asymptotically dominant term. The two terms in the large bracket are nicely combined upon putting $k=\omega_3/v$ to yield:

$$E^{(2)}(x,t)=-Ae^{-i\omega_3 t}\,\frac{2x-L_1-L_2}{\omega_1\omega_2}\int_{-\infty}^{\infty}\frac{dv}{v}\frac{df_0}{dv}$$

$$\frac{exp\,[i\frac{\omega_1}{v}L_1-i\frac{\omega_2}{v}L_2+i\frac{\omega_3}{v}x]}{\varepsilon(\frac{\omega_3}{v},\omega_3)\varepsilon(\frac{\omega_2}{v},\omega_2)\varepsilon(\frac{-\omega_1}{v},-\omega_1)} \tag{44}$$

Echo is given rise to where the exponential argument becomes zero, thus the velocity integral is not phase-mixed:

$$x_{echo} = \frac{\omega_2 L_2 - \omega_1 L_1}{\omega_3} \tag{45}$$

where we have:

$$E^{(2)}(x_{echo},t) = Ae^{-i\omega_3 t} \frac{(\omega_1+\omega_2)(L_1-L_2)}{\omega_1\omega_2\omega_3} \int_{-\infty}^{\infty} \frac{dv}{v} \frac{\frac{df_0}{dv}}{\varepsilon(\frac{\omega_3}{v},\omega_3)\varepsilon(\frac{\omega_2}{v},\omega_2)\varepsilon(\frac{-\omega_1}{v},-\omega_1)} \tag{46}$$

The above echo given by Eqs. (45) and (46) can be given rise to in a semi-bounded plasma as well as in an infinite plasma [2].

Repeating similar algebra performed in the foregoing analysis, we obtain from Term Eq. (35) an echo occurrence at:

$x_{echo} = \frac{L_1}{2}$, and

$$E^{(2)}(x_{echo},t) = -\alpha\rho_1^2 \frac{eL_1}{8\pi m} \frac{\omega_p^2}{\omega_1^2} e^{2i\omega_1 t} \int_{-\infty}^{\infty} dv \frac{\frac{1}{v}\frac{df_0}{dv}}{\varepsilon(-\frac{2\omega_1}{v},-2\omega_1)\varepsilon(-\frac{\omega_1}{v},-\omega_1),\varepsilon(-\frac{\omega_1}{v},-\omega_1)} \tag{47}$$

We obtain from Term Eq. (36) an echo occurrence at: $x_{echo} = \frac{L_2}{2}$, and

$$E^{(2)}(x_{echo},t) = -\alpha\rho_2^2 \frac{eL_2}{8\pi m} \frac{\omega_p^2}{\omega_2^2} e^{-2i\omega_2 t} \int_{-\infty}^{\infty} dv \frac{\frac{1}{v}\frac{df_0}{dv}}{\varepsilon(\frac{2\omega_2}{v},2\omega_2)\varepsilon(\frac{\omega_2}{v},\omega_2),\varepsilon(\frac{\omega_2}{v},\omega_2)} \tag{48}$$

Term Eq. (37) gives an echo occurrence at: $x_{echo} = -\frac{L_1\omega_1}{\omega_3}$ $(\omega_3 = \omega_2 - \omega_1)$, and

$$E^{(2)}(x_{echo},t) = \alpha\rho_1\rho_2 \frac{eL_1}{4\pi m} \frac{\omega_p^2}{\omega_1\omega_3} e^{-i\omega_3 t} \int_{-\infty}^{\infty} dv \frac{\frac{1}{v}\frac{df_0}{dv}}{\varepsilon(\frac{-\omega_1}{v},-\omega_1)\varepsilon(\frac{\omega_2}{v},\omega_2)\varepsilon(\frac{\omega_3}{v},\omega_3)} \tag{49}$$

Term (38) gives an echo occurrence at:

$x_{echo} = \frac{L_2\omega_2}{\omega_3}$, and

$$E^{(2)}(x_{echo},t) = -\alpha\rho_1\rho_2 \frac{eL_2}{4\pi m} \frac{\omega_p^2}{\omega_2\omega_3} e^{-i\omega_3 t} \int_{-\infty}^{\infty} dv \frac{\frac{1}{v}\frac{df_0}{dv}}{\varepsilon(\frac{-\omega_1}{v},-\omega_1)\varepsilon(\frac{\omega_2}{v},\omega_2)\varepsilon(\frac{\omega_3}{v},\omega_3)} \tag{50}$$

5. DISCUSSION

In dealing with Eq. (8), some authors [4, 5] directly solved the equation which is a first order linear differential equation. Here we show the mathematical equivalence of this approach with that presented in section III. Eq. (8) can be solved in the form:

$$f(x,v) = e^{i\frac{\omega}{v}x}\left[f(x_0,v)e^{-i\frac{\omega}{v}x_0} + \int_{x_0}^{x} dx' \frac{e}{mv} E(x') \frac{df_0}{dv} e^{-i\frac{\omega}{v}x'} \right] \tag{51}$$

where x_0 is the reference point of the boundary condition, and $\omega(=\omega_r + i\varepsilon)$ is assumed to have a small infinitesimal positive imaginary part. This implicit condition is necessary to have a legitimate time-Fourier transform of $f(t)$.

i) For $v<0$: let us take the reference point $x_0 \to \infty$. Then the exponential factor in the first term in the large bracket [ ] vanishes owing to $\varepsilon > 0$, and the solution can be written as:

$$f(x,v) = e^{i\frac{\omega}{v}x} \int_{\infty}^{x} dx' \frac{e}{mv} E(x') \frac{df_0}{dv} e^{-i\frac{\omega}{v}x'} \equiv f_N(x,v)\,(v<0) \tag{52}$$

ii) For $v>0$: let us take $x_0 = 0$ in Eq. (51) to write:

$$f(x,v) = e^{i\frac{\omega}{v}x}\left[f(0,v) + \int_0^x dx' \frac{e}{mv} E(x') \frac{df_0}{dv} e^{-i\frac{\omega}{v}x'} \right]$$

To enforce the boundary condition Eq. (4), we use Eq. (52) to write:

$$f(0,v) = f(0,-v) = -\int_0^{\infty} dx' \frac{e}{mv} E(x') \frac{df_0}{dv} e^{i\frac{\omega}{v}x'}$$

because $f_0(v)$ is an even function of v. Therefore we obtain:

$$f(x,v) = e^{i\frac{\omega}{v}x}\left[-\int_0^{\infty} dx' \frac{e}{mv} E(x') \frac{df_0}{dv} e^{i\frac{\omega}{v}x'} + \int_0^{x} dx' \frac{e}{mv} E(x') \frac{df_0}{dv} e^{-i\frac{\omega}{v}x'} \right] \equiv f_P(x,v) \quad (v>0) \tag{53}$$

Eqs. (52) and (53) were derived by Landau [4] in his analysis of electric field in a semi-infinite plasma. Note that the solutions satisfy the boundary condition at $x=\infty$ i.e. $f(v, x=\infty)=0$ and the specular reflection condition.

It can be easily shown that both functions $f_N(x,v)$ and $f_P(x,v)$ are Fourier transformed, upon using the extension $E(-x) = -E(x)$, in the same form:

$$f_{N,P}(k,v) = \frac{ie}{m}\frac{\frac{df_0}{dv}}{\omega - kv}E(k) \tag{54}$$

It is instructive to show that the inverse transform of Eq. (54) recovers Eqs. (52) and (53), respectively. We proceed as follows:

$$f(x,\omega,v) = \frac{ie}{m}\frac{df_0}{dv}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}}{\omega - kv}E(k,\omega) \tag{55}$$

Putting the Fourier expression, $E(k) = \int_{-\infty}^{\infty}E(\xi)e^{-ik\xi}d\xi$, in the above, we write:

$$f(x,\omega,v) = -\frac{1}{2\pi}\frac{ie}{mv}\frac{df_0}{dv}\int_{-\infty}^{\infty}d\xi E(\xi)\int_{-\infty}^{\infty}dk\frac{e^{ik(x-\xi)}}{k-\frac{\omega}{v}} \tag{56}$$

Here we should assume ω to have an infinitesimal positive imaginary part (ε), and let $\varepsilon \to 0$. The singularity $k = \frac{\omega}{v}$ in the $\int dk$ --integral now approaches the real k --axis from above (from below) if $v > 0$ ($v < 0$) in the complex k --plane. Therefore, the path of the integral, the real k --axis, should be deformed into the principal part plus the concave-down (convex-up) small half-circle going around the singularity $k = \frac{\omega}{v}$ in the positive direction (negative direction) when $v < 0$ ($v < 0$). The deformed contours are the analytic continuations of the $\int dk$ --integral when the singularity is located on the real k --axis. Thus we write:

$$\int_{-\infty}^{\infty}dk\frac{e^{ik(x-\xi)}}{k-\frac{\omega}{v}} = P\int_{-\infty}^{\infty}dk\frac{e^{ik(x-\xi)}}{k-\frac{\omega}{v}} \pm i\pi\, e^{i\frac{\omega}{v}(x-\xi)}$$

where +(--) sign corresponds to $v > 0\,(v < 0)$. As for the principal part integral, we have the formula [6]:

$$P\int_{-\infty}^{\infty}dk\frac{e^{ik(x-\xi)}}{k-\frac{\omega}{v}} = i\pi\, e^{i\frac{\omega}{v}(x-\xi)}S(x-\xi)$$

where S is the step function.

Now Eq. (56) takes different values depending upon whether $v > 0$ or $v < 0$. When $v < 0$, it can be straightly shown that Eq. (56) takes the form:

$$f_N(x,\omega,v) = \frac{e}{mv}\frac{df_0}{dv}\int_{\infty}^{x}d\xi E^{(1)}(\xi)\, e^{i\frac{\omega}{v}(x-\xi)} \quad (v < 0) \tag{57}$$

When $v > 0$, f in Eq. (56) takes the form:

$$f_P(x,\omega,v) = \frac{e}{mv}\frac{df_0}{dv}\int_{-\infty}^{x}d\xi E(\xi)\, e^{i\frac{\omega}{v}(x-\xi)} \quad (v > 0) \tag{58}$$

The above equation can be put into Eq. (53) upon using $E(-\xi) = -E(\xi)$. Eqs. (53) and (57) are the Fourier inversion of Eq. (54) whose form is valid for unbounded plasma as well as for semi-bounded plasma. Then, what difference does a semi-bounded plasma make? The sole mathematical distinction is that $E(x)$ is artificially defined as an odd function in a semi-bounded plasma while there is no such restriction for $E(x)$ in unbounded plasma.

In this work, we determined echo solutions by calculating the second order electric field in response to the external charges as prescribed in Eq. (3). We paid special attention to the electric boundary condition at the interface in regard to its connection with the surface echoes. The terms E_0 and '1′ in the factors $(1+e^{-ikL_1})$ and $(1+e^{-ikL_2})$ in Eq. (23) are derived from the boundary condition. The surface boundary terms generate echoes in diversity of spots. Thus, it appears that the boundary condition-related echoes should receive more attention.

Earlier, Sitenko *et al.* [5] investigated surface echo in a semi-bounded plasma. Their result had no difference from the result for an infinite plasma. Their Eq. (5) in Ref. [5] for the electric field of a semi-bounded plasma has no difference from that of an infinite plasma. Their analysis can begin with Eq. (17) in this work with $N^{(1)} = 0$ or the last equation of section III. This omission results in failure to locate the extra echo spots that occur because of the boundary. $N^{(1)}$ cannot be zero in a plasma with a boundary. This important quantity stems from the presence of the vacuum-plasma boundary. Vlasov equation is solved in a semi-bounded plasma with oddly continued electric field, and the Poisson equation gives rise to $N^{(1)} \neq 0$ because of the discontinuity of $E(x)$ at $x = 0$. This is also ascertained by directly solving Vlasov equation as discussed in the earlier part of this section.

It was reported [7] that an exact nonlinear solution of surface wave in a bounded plasma impressed by external charges was obtained. Echoes associated with surface waves in a semi-bounded plasma or a plasma slab needs further investigation. The linear waves in a bounded plasma which are the starting basis to investigate the second order quantities have been comprehensively dealt with in Ref. [8]. Kinetic solutions of plasma waves in a slab were investigated by Lee and Lim [9]. The above references would be useful for considering higher order solutions generated in response to external charges in a slab or semi-infinite plasma.

CONFLICT OF INTEREST

The author confirms that this article content has no conflict of interest.

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Declare none.

REFERENCES

[1] Gould RW, O'Neil TM, Malmberg JH. Plasma wave echo. Phys. Rev Lett 1967; 19: 219-22.

[2] Krall NA, Trivelpiece AW. Principles of Plasma Physics. New York: McGraw-Hill 1973; p. 547.

[3] Malmberg JH, Wharton CB, Gould RW, O'Neil TM. Plasma wave echo experiment. Phys Rev Lett 1968; 20: 95.

[4] Landau L. On the vibration of the electronic plasma. J Phys 1946; 10: 25-34.

[5] Sitenko AG, Pavlenko VN, Zasenko VI. Echo in a half-space plasma. Phys Lett 1975; 53A: 259-60.

[6] Arfken G. Mathematical Methods for Physicists. Orlando: Academic Press 1985; p. 415.

[7] Stenflo L, Yu MY. An exact solution for externally driven surface plasma waves. Phys Plasmas 2002; 9: 5129-30.

[8] Gradov OM, Stenflo L, Linear theory of a cold bounded plasm. Phys Rep 1983; 3: 111-37.

[9] Lee HJ, Lim Yk. Kinetic theory of surface waves in a plasma slab. J Korean Phys Soc 2007; 50: 1056-61.

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