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Two-Weight Orlicz Type Integral Inequalities for the Maximal Operator[1]

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Abstract: We present a two-weight Orlicz-type integral inequality for the maximal operator which characterizes $(u,v)\in A_p$.

Keywords: Maximal operator, two-weights.

1. INTRODUCTION

In this paper we will study integral inequalities of the type

$$\int_{R^n}\Phi(Mf(x)^p)u(x)dx \le c_1\int_{R^n}\Psi(c_2\,|f(x)|^p)v(x)dx, \qquad (1)$$

where $Mf(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(t)|\,dt$ is the Hardy-Littlewood maximal operator, and we ask for condtions on $\Phi,\Psi:R_+\to R_+$ such that (1) holds if and only if $(u,v)\in A_p$.

We say that $(u,v)\in A_p$ if $\frac{1}{|Q|}\int_Q u\left(\frac{1}{|Q|}\int_Q v^{1-p'}\right)^{p-1}\le c<\infty, 1<p<\infty$, and $Mu(x)\le cv(x)$, if $p=1$. These weight classes were introduced by Muckenhoupt [4] and Muckenhoupt and Wheeden [5] to study (1) when $\Phi(t)=\Psi(t)=t$. If $1<p<\infty$ and $u=v\in A_p$, (1) holds for $\Phi(t)=\Psi(t)=t$, but not if $p=1$. Also for each $1\le p<\infty$ there exists a pair $(u,v)\in A_p$ so that (1) fails in the special case $\Phi(t)=\Psi(t)=t$ [3, p. 395]. In these exceptional cases we have a weak type inequality. An excellent reference is the book by J.Garcia-Cuerva and J.L.Rubio de Francia [3]. We refer the reader interested in the current state of the two-weight theory to the recent book [1] by Cruz-Uribe, Martell, and Pérez.

The restrictions on Φ,Ψ are: $\Phi(t)=\int_0^t a(s)ds, \Psi(t)=\int_0^t b(s)ds$ with $a,b:R_+\to R_+$ satisfying

$$\int_0^s \frac{a(t)}{t}dt \le c'b(c''s), 0<s<\infty. \qquad (*)$$

Note that this excludes the classical case $\Phi(t)=\Psi(t)=t$. If $(*)$ holds, we say that Φ,Ψ are (c',c'')-related.

We are now ready to state our main result whose proof will be given in section 3.

Theorem 1 The following statements are equivalent for $1\le p<\infty$.

(2) For each Φ and Ψ which are (c',c'')-related, we have

$$\int_{R^n}\Phi(Mf^p)u \le c_1\int_{R^n}\Psi(c_2\,|f|^p)v,$$

for all $f:R^n\to R$, where the constamts c_1,c_2 depend only on c',c'' and p.

(3) We have $(u,v)\in A_p$.

Remark: In the Lebesgue measure case - $u=v=1$ - integral inequalities related to (2) can be found in [6]. It should be noted that $p=1$ is not excluded.

In section 4 we will examine in what sense the condition $\int_0^t \frac{a(s)}{s}ds \le c'b(c''t)$ is also necessary for Theorem 1, and in section 5 we will examine the extrapolation problem: when is it possible to replace p by $p-\varepsilon$ in (2). In sections 6 and 7 we will study the iterated maximal operator and its relation to extrapolation. In section 8 we will collect some unusual and surprising integral inequalities for Mf obtained by choosing Φ,Ψ and applying Theoren 1.

A final comment is in order. I have dedicated this paper to the memory of Richard A. Hunt who made significant contributions to the theory of A_p-weights and to whom I am indebted for introducing me to this subject some 40 years ago.

2. A TWO-WEIGHT DISTRIBUTIONAL INEQUALITY

For convenience all our functions will be non-negative: $f:R^n\to R_+$.

The distributional inequality below for $u=v=1$ - the Lebesgue measure case - and a sublinear operator T instead of M is equivalent with saying that T is both weak-type (p,p) and of type (∞,∞) [11, p. 103].

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Theorem 2 The following statements are equivalent for $1 \le p < \infty$.

(4) There exists $0 < c_0 < \infty$ such that for every $f: R_n \to R_+$ we have for $0 < t < \infty$

$$u\{x: Mf(x) > t\} \le \frac{c_0}{t^p} \int_{t/c_0}^{\infty} v\{x: f(x) > s\} s^{p-1} ds.$$

(5) We have $(u,v) \in A_p$.

Proof. Apart from a minor detail, the proof follows the standard covering argument and we include it for the benefit of the reader.

$(5) \to (4)$. We may assume that M is the centered maximal operator

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q f(\tau) d\tau,$$

where the $\sup$ is extended over all cubes Q centered at x. We consider the case $1 < p < \infty$ first. Fix $f: R_n \to R_+$, and for $0 < t < \infty$ let $f = f^t + f_t$, where

$$f^t(x) = \begin{cases} 0, & f(x) \le t/2 \\ f(x), & f(x) > t/2. \end{cases}$$

Then $Mf(x) \le Mf^t(x) + Mf_t(x)$ so that $\{Mf > t\} \subset \{Mf^t > t/2\} \equiv E_t$. Let $E_{tN} = E_t \cap \{x: |x| \le N\}$. We can now apply the Besicovitch covering Theorem and obtain cubes $\{Q_j\}$ satisfying

$$E_{tN} \subset \cup Q_j, |Q_j| \le \frac{2}{t} \int_{Q_j} f^t, \sum \chi_{Q_j} \le c < \infty.$$

Then

$$u(E_{tN}) \le \sum u(Q_j) \le \frac{c}{t^p} \sum \frac{u(Q_j)}{|Q_j|^p} \left(\int_{Q_j} f^t v^{1/p} v^{-1/p} \right)^p$$

$$\le \frac{c}{t^p} \sum \frac{u(Q_j)}{|Q_j|^p} \int_{Q_j} (f^t)^p v \cdot \left(\int_{Q_j} v^{1-p'} \right)^{p-1}$$

$$\le \frac{c}{t^p} \int_{\{f \ge t/2\}} f^p v.$$

If $A_t = \{x: f(x) \ge t/2\}$, then

$$u(E_{tN}) \le \frac{c}{t^p} \int_{R^n} (f\chi_{A_t})^p v = \frac{c}{t^p} \int_0^{\infty} v\{f\chi_{A_t} > s\} s^{p-1} ds$$

$$= \frac{c}{t^p} \left(\int_{t/2}^{\infty} v\{f > s\} s^{p-1} ds + v(A_t) \int_0^{t/2} s^{p-1} ds \right).$$

It is clear that for some constant c

$$c \int_{t/4}^{t/2} v\{f > s\} s^{p-1} ds \ge v(A_t) \int_0^{t/2} s^{p-1} ds,$$

and hence for some constant c_0

$$u(E_{tN}) \le \frac{c_0}{t^p} \int_{t/c_0}^{\infty} v\{f > s\} s^{p-1} ds.$$

Let now $N \to \infty$. We use the same notation for the case $p = 1$ as above. Since now $u(Q_j)/|Q_j| \le \inf_{Q_j} v$ we get

$$u(E_{tN}) \le \frac{c}{t} \sum \frac{u(Q_j)}{|Q_j|} \int_{Q_j} f^t$$

$$\le \frac{c}{t} \sum \int_{Q_j} f^t v \le \frac{c}{t} \int_{R^n} f\chi_{A_t} v.$$

Proceed now as in the case $1 < p < \infty$.

$(4) \to (5)$. For the case $p = 1$ we fix a cube Q_0 and let $f = \chi_Q$, where Q is an arbitrary subcube of Q_0. Then

$$Q_0 \subset \left\{ Mf \ge \frac{1}{|Q_0|} \int_Q f = \frac{|Q|}{|Q_0|} \equiv t \right\}.$$

Thus $u(Q_0) \le c_0(|Q_0|/|Q|) v(Q)$, and thus $u(Q_0)/|Q_0| \le c_0 \inf_{Q_0} v$.

If $1 < p < \infty$ we take the usual test function $f = \chi_Q v^{1-p'}$ with $t = \frac{1}{|Q|} \int_Q f$. Then

$$u(Q) \le c_0 \frac{|Q|^p}{\left(\int_Q f \right)^p} \int_{t/c_0}^{\infty} v\{f > s\} s^{p-1} ds$$

$$\le c_0 \frac{|Q|^p}{\left(\int_Q f \right)^p} \int_Q f^p v$$

$$= c_0 |Q|^p \left(\int_Q v^{1-p'} \right)^{1-p},$$

and the A_p-condition follows.

3. PROOF OF THEOREM 1. (3) $\to$ (2).

$$\int_{R^n} \Phi[Mf(x)^p] u(x) dx = \int_0^{\infty} u\{Mf^p > t\} a(t) dt =$$

$$\int_0^{\infty} u\{Mf > t^{1/p}\} a(t) dt \le c_0 \int_0^{\infty} \frac{1}{t} \int_{t^{1/p}/c_0}^{\infty} v\{f > s\} s^{p-1} ds\, a(t) dt =$$

$$c_0 \int_0^{\infty} \int_0^{(c_0 s)^p} \frac{a(t)}{t} v\{f > s\} s^{p-1} dt ds \le$$

$$c_0 c' \int_0^{\infty} b(c''(c_0 s)^p) v\{f > s\} s^{p-1} ds =$$

$$\frac{c_0 c'}{p} \int_0^{\infty} b(c_* t) v\{f^p > t\} dt \le c_1 \int_{R^n} \Psi[c_2 f(x)^p] v(x) dx.$$

It is clear that the constants c_1 and c_2 have the desired properties.

(2) $\to$ (3). We assume that

$$L \equiv \int_0^\infty u\{Mf^p > t\}a(t)dt \le c_1\int_0^\infty v\{c_2 f^p > t\}b(t)dt \equiv R.$$

Fix $0 < \lambda_0 < \infty$ and let

$$a(t) = 1h\chi_{[\lambda_0,\lambda_0+h]}(t).$$

Set

$$b(t) = \int_0^t a(s)sds = 0, 0 \le t \le \lambda_0 1h\log(t/\lambda_0), \lambda_0 < t \le \lambda_0 + h1h$$
$$\log\lambda_0 + h\lambda_0, t > \lambda_0 + h.$$

With this choice Φ and Ψ are $(1,1)$-related independent of h and λ_0 and hence c_1 and c_2 do not depend on h or λ_0. Then

$$L = 1h\int_{\lambda_0}^{\lambda_0+h} u\{Mf^p > t\}dt \to u\{Mf^p > \lambda_0\},$$

as $h \to 0$. The right side R is

$$R = c_1 h\int_{\lambda_0}^{\lambda_0+h} v\{c^2 f^p > t\}\log(t/\lambda_0)dt + c_1 h\log\lambda_0 + h\lambda_0$$
$$\int_{\lambda_0+h}^\infty v\{c_2 f^p > t\}dt = I_1(h) + I_2(h).$$

We see that $I_1(h) \to 0$ as $h \to 0$ and

$$I_2(h) \to c_1\lambda_0\int_{\lambda_0}^\infty v\{c_2 f^p > t\}dt = c_1 c_2\lambda_0\int_{\lambda_0/c_2}^\infty v\{f^p > t\}dt.$$

Since λ_0 was arbitrary we get for some constant $c_0 > 1$

$$u\{Mf^p > \lambda\} \le c_0\lambda\int_{\lambda/c_0}^\infty v\{f^p > t\}dt.$$

We now make the substitution $\lambda = s^p$ and then $t \to t^p$ to get

$$u\{Mf > s\} \le c_0 s^p\int_{s/c_0}^\infty v\{f > t\}t^{p-1}dt.$$

By Theorem 2 this is the same as saying $(u,v) \in A_p$.

Remark. Theorem 1 is not true with M replaced by a singular integral operator T. If it were true, then the argument as on the previous page shows that

$$u\{|Tf| > s\} \le c_0 s^p\int_{s/c_0}^\infty v\{f > t\}t^{p-1}dt,$$

and hence for $s > c_0\|f\|_{\infty,v}, u\{|Tf| > s\} = 0$ and $\|Tf\|_{\infty,u} < \infty$. But T is not of type (∞,∞) [10].

4. A CONVERSE

For a given $a, b : R_+ \to R_+$ and $\Phi(t) = \int_0^t a(s)ds$, $\Psi(t) = \int_0^t b(s)ds$ we wish to examine when (2) of Theorem 1 implies that

$$\int_0^s a(t)tdt \le c' b(c'' s), 0 < s < \infty.$$

Since this condition is independent of $(u,v) \in A_p$, we are allowed to take any $(u,v) \in A_p$, in particular $u = v = 1$, the Lebesgue measure case, or $u = v$ in RH_∞. We prefer the second alternative since it is based on an extension of the reverse weak type inequality. We say that $u \in RH_\infty$ if for every cube Q, $\sup_Q u(x) \le c1|Q|\int_Q u$. The inf of all such c's is called the RH_∞-constant of u. This class was studied in [2] and plays roughly the same role among the reverse Hölder classes $RH_r, r \to \infty$, as A_1 does among A_p, $p \searrow 1$. Typical examples of RH_∞-weights in R_+ are $u(x) = x^\alpha$, $\alpha > 0$.

Theorem 3 Let $u \in RH_\infty$. Then there are constants $0 < c_1, c' < \infty$ such that for all $f : R^n \to R_+$ and $0 < t < \infty$

$$1t\int_{\{f>t\}} f(x)u(x)dx \le c_1 u\{Mf > c't\},$$

where $1/c' = c_*$ is the RH_∞-constant of u.

Proof. Since $u(x)dx$ is a doubling measure [3], we have available the Calderon-Zygmund decomposition at height t and this gives us disjoint cubes $\{Q_k\}$ such that

$$t \le 1u(Q_k)\int_{Q_k} fu \le ct$$

$$f(x) \le t, \text{ on } R^n \setminus \cup Q_k.$$

Then

$$1t\int_{\{f>t\}} fu \le 1t\sum\int_{Q_k} fu \le c\sum u(Q_k) = cu(\cup Q_k) \le cu\{M_u f > t\},$$

where $M_u f(x) = \sup_{x\in Q} 1u(Q)\int_Q fu$. Since $u \in RH_\infty$

$$1u(Q)\int_Q fu \le \sup_Q uu(Q)/|Q|1|Q|\int_Q f \le c_* Mf(x),$$

if $x \in Q$. Hence $m_u f(x) \le c_* Mf(x)$ and the proof is complete.

Defintion. (1) $b : R_+ \to R_+$ is quasi-increasing (qi) if there is a constant $0 < c_0 < \infty$ such that $t' \le t''$ implies $b(t') \le c_0 b(c_0 t'')$.

(2) A measure μ on R_+ is weakly doubling if there is a constant $0 < c < \infty$ such that $\mu([0,2d]) \le c\mu([d,2d]), 0 < d < \infty$.

If a measure is doubling, it is also weakly doubling. The converse is not true as the measure $d\mu = e^x dx$ shows. In fact if

$f:R_+ \to R_+$ is nondecreasing, then $d\mu = f(x)dx$ is weakly doubling. The measure $d\mu = dx/(1+x)$ is not weakly doubling.

Theorem 4 Assume that $b(t)$ is qi and assume that for some n and $u_0 \in RH_\infty(R^n)$ we have

$$\int_{R^n} \Phi(Mf^p)u_0 \le c_1 \int_{R^n} \Psi(c_2 f^p)u_0.$$

Then

$$\int_0^s a(t)tdt \le c'b(c''s), 0 < s < \infty$$

holds if $p=1$, and if $1<p<\infty$ it holds under the additional assumption that the measure $d\mu = a(t)tdt$ is weakly doubling.

Proof. In distributional form the integral inequality is

$$L \equiv \int_0^\infty u_0\{Mf^p > t\}a(t)dt \le c_1 \int_0^\infty u_0\{c_2 f^p > t\}b(t)dt \equiv R.$$

The constants $c_1, c_2, \ldots$ appearing below only depend upon the constants in the overall hypothesis. By Lemma 3

$$L \ge c_3 \int_0^\infty a(t)t^{1/p} \int_{\{f > c_4 t^{1/p}\}} f(x)u_0(x)dxdt.$$

We apply this to the test functions $f(x) = r\chi_Q(x)$, $0<r<\infty$, $Q=[0,1]^n$ and get

$$L \ge c_3 \int_0^{c_5 r^p} a(t)t^{1/p} r u_1 dt, u_1 = \int_Q u_0(x)dx.$$

The right side $R = \int_0^{c_6 r^p} u_1 b(t)dt$. Hence

$$c_3 r \int_0^{c_5 r^p} a(t)t^{1/p}dt \le c_1 \int_0^{c_6 r^p} b(t)dt.$$

With $s = c_5 r^p$ this becomes

$$c_7 s^{1/p} \int_0^s a(t)t^{1/p}dt \le c_1 \int_0^{c_8 s} b(t)dt \le c_9 sb(c_{10}s),$$

since b is quasi-increasing. The left side is

$$\ge c_7 s^{1/p} \int_{s/2}^s t^{1/p'} a(t)tdt \ge c_{11} s \int_0^s a(t)tdt,$$

by the weak type doubling condition, which clearly is not needed when $p=1$.

Remark: 1. The special case $p=1$ and $u_0 \sim 1$ - the Lebesgue measure case - is Theorem 7 in [6].

2. The weak doubling hypothesis of the measure $d\mu = a(t)tdt$ cannot be omitted if $1<p<\infty$. The classical norm inequality for $u \in A_p$ is

$$\int_{R^n} Mf^p u \le c \int_{R^n} f^p u.$$

This is the $\Phi(t) = \Psi(t) = t$ case, and $a(t)=1$.

5. EXTRAPOLATION

As before $a,b: R_+ \to R_+$ and $\Phi(s) = \int_0^s a(t)dt$, $\Psi(s) = \int_0^s b(t)dt$. We wish to examine the relationship between the following statements.

I. There exists $0<\varepsilon<p, 1\le p<\infty$, such that for $(u,v) \in A_p$ we have

$$\int_{R^n} \Phi(Mf^{p-\varepsilon})u \le c_1 \int_{R^n} \Psi(c_2 f^{p-\varepsilon})v.$$

II. There exists $\eta > 0$ such that

$$\int_0^s a(t)t^{1+\eta}dt \le c'b(c''s)s^\eta, 0<s<\infty.$$

The constants ε, η, and p are related by $\varepsilon = \eta p/(1+\eta)$ or $\eta = \varepsilon/(p-\varepsilon)$.

Theorem 5 $II \Rightarrow I$, and, if b is quasi-increasing and $u=v=1$, the converse $I \Rightarrow II$ holds if $p=1$, and if $1<p<\infty$ it holds if the measure $d\mu = a(t)/t^{1+\eta}dt$ is weakly doubling.

Proof. $II \Rightarrow I$. Fix $1 \le p < \infty$ and let $\varepsilon = \eta p/(1+\eta)$. If $q = p-\varepsilon$, then $p/q = 1+\eta$. By Theorem 2

$$\int_{R^n} \Phi(Mf^q)u \le c_0 \int_0^\infty a(t)t^{1+\eta} \int_{t^{1/q}/c_0}^\infty v\{f>s\}s^{p-1}dsdt = c_0 \int_0^\infty \int_0^{(c_0 s)^q} a(t)t^{1+\eta}dt v\{f>s\}s^{p-1}ds \le c\int_0^\infty b[(c_0' s)^q](c_0 s)^{q\eta} v\{f>s\}s^{p-1}ds = c\int_0^\infty b(\sigma)\sigma^\eta v\{f>\sigma^{1/q}\}\sigma^{(p-1)/q}\sigma^{1/q-1}d\sigma = c\int_0^\infty v\{f>\sigma^{1/q}\}b(\sigma)d\sigma = c_1 \int_{R^n} \Psi(c_2 f^q)v.$$

$I \Rightarrow II$. First let $p=1$ and $u=v=1$. If $q=1-\varepsilon$, then the statement I in distributional form is

$$L = \int_0^\infty |\{Mf > t^{1/q}\}| a(t)dt \le c_1 \int_0^\infty |\{f > c_3 t^{1/q}\}| b(t)dt = R.$$

By Lemma 3,

$$L \ge c_4 \int_0^\infty a(t)t^{1/q} \int_{\{f > c_5 t^{1/q}\}} f(x)dxdt.$$

We apply this to the test functions $f(x) = r\chi_{[0,1]}(x)$, $0<r<\infty$. Then

$$L \ge c_4 \int_0^{c_6 r^q} ra(t)t^{1/q}dt, R = \int_0^{c_7 r^q} b(t)dt \le c_8 r^q b(c_9 r^q),$$

because b is quasi-increasing. Hence

$$\int_0^{c_6 r^q} a(t)t^{1/q}dt \le c_{10} r^{q-1} b(c_9 r^q).$$

Let $s = c_6 r^q$ and $1/q = 1+\eta$. Then $\eta = \varepsilon/(1-\varepsilon)$ and

$$\int_0^s a(t)t^{1+\eta}dt \le c_{11} s^{(q-1)/q} b(c_{12}s),$$

and $(q-1)/q=-\eta$.

The case $1<p<\infty$ with $q=p-\varepsilon$, and $u=v=1$ follows the same steps as above and we get

$$\int_0^s \frac{a(t)}{t^{1/q}}dt \le c_{11}s^{(q-1)/q}b(c_{12}s).$$

We use now the weak doubling condition and get

$$\int_0^s \frac{a(t)}{t^{1/q}}dt=\int_0^s \frac{a(t)t^{1+\eta-1/q}}{t^{1+\eta}} \ge c_{13}s^{1+\eta-1/q}\int_{s/2}^s \frac{a(t)}{t^{1+\eta}}dt \ge c_{14}s^{1+\eta-1/q}\int_0^s \frac{a(t)}{t^{1+\eta}}dt.$$

Hence

$$\int_0^s \frac{a(t)}{t^{1+\eta}}dt \le c_{15}b(c_{12}s)/s^{\eta}.$$

The result that we discuss now essentially says that, in the presence of condition II, extrapolation for (u,v) is the same as $(u,v)\in A_p$.

Theorem 6 Let $1\le p<\infty, \eta\ge 0, \varepsilon=\eta p/(1+\eta)$, and

$$\int_0^s \frac{a(t)}{t^{1+\eta}}dt \le \frac{c'b(c''s)}{s^{\eta}}, 0<s<\infty.$$

Then the following statements are equivalent.

$$\int_{R^n}\Phi(Mf^{\,p-\varepsilon})u \le c_1\int_{R^n}\Psi(c_2f^{\,p-\varepsilon})v, \qquad (2)$$

where c_1, c_2 depend only upon c', c'' and p.

(2) We have $(u,v)\in A_p$.

Remark: Theorem 1 is the special case $\eta=0$.

Proof. $(2)\Rightarrow(1)$. This is $II\Rightarrow I$ of Theorem 5. $(1)\Rightarrow(2)$. We proceed as in the proof of Theorem 1 and let

$$a(t)=\frac{1}{h}\chi_{[\lambda,\lambda+h]}(t), \lambda>0, h>0.$$

We let $b(s)=s^{\eta}\int_0^s \frac{a(t)}{t^{1+\eta}}dt$. We may assume that $\eta>0$ since the case $\eta=0$ is Theorem 1. Then

$$b(s)=0, 0\le s\le\lambda \frac{(s/\lambda)^{\eta}-1}{h\eta}, \lambda\le s\le\lambda+h \frac{(s/\lambda)^{\eta}-(s/(\lambda+h))^{\eta}}{h\eta}, s\ge\lambda+h.$$

Our hypothesis in distributional form is

$$L_h \equiv \int_0^{\infty} u\{Mf^{\,p-\varepsilon}>t\}a(t)dt \le c_1\int_{\lambda}^{\infty} v\{c_2f^{\,p-\varepsilon}>t\}b(t)dt \equiv R_h.$$

First

$$L_h=\frac{1}{h}\int_{\lambda}^{\lambda+h} u\{Mf^{\,p-\varepsilon}>t\}dt \to u\{Mf^{\,p-\varepsilon}>\lambda\},$$

as $h\to 0$. The right side R_h splits into two integrals

$$R_h=c_1\left(\int_{\lambda}^{\lambda+h}+\int_{\lambda+h}^{\infty}\right)=I_1+I_2.$$

I_1 is easily disposed of

$$I_1=c_1\int_{\lambda}^{\lambda+h}\frac{(t/\lambda)^{\eta}-1}{h\eta}v\{c_2f^{\,p-\varepsilon}>t\}dt \to 0,$$

as $h\to 0$. Next

$$I_2=c_1\frac{\lambda^{-\eta}-(\lambda+h)^{-\eta}}{h\eta}\int_{\lambda+h}^{\infty} v\{c_2f^{\,p-\varepsilon}>t\}t^{\eta}dt \to \frac{c_1}{\lambda^{\eta+1}}\int_{\lambda}^{\infty} v\{c_2f^{\,p-\varepsilon}>t\}t^{\eta}dt,$$

as $h\to 0$. The substitution $\tau=t^{\eta+1}$ gives

$$I_2 \to \frac{c_3}{\lambda^{\eta+1}}\int_{\lambda^{\eta+1}}^{\infty} v\{c_2f^{\,p-\varepsilon}>\tau^{1/(\eta+1)}\}d\tau,$$

and since $(p-\varepsilon)(\eta+1)=p(\eta+1)-p\eta=p$,

$$I_2 \to \frac{c_3}{\lambda^{\eta+1}}\int_{\lambda^{\eta+1}}^{\infty} v\{c_4f^{\,p}>\tau\}d\tau.$$

Hence for some constant $c_0>1$

$$u\{Mf^{\,p-\varepsilon}>\lambda\} \le \frac{c_0}{\lambda^{\eta+1}}\int_{\lambda^{\eta+1}/c_0}^{\infty} v\{f^{\,p}>t\}dt.$$

With $\lambda=\sigma^{p-\varepsilon}$ we get

$$u\{Mf>\sigma\} \le \frac{c_0}{\sigma^p}\int_{\sigma^p/c_0}^{\infty} v\{f^{\,p}>t\}dt=\frac{c_{0'}}{\sigma^p}\int_{\sigma/c_{0'}}^{\infty} v\{f>t\}t^{p-1}dt.$$

This shows that $(u,v)\in A_p$ by Theorem 2.

Remark: The following observation may be of interest in connection with condition II: if $\int_0^s \frac{a(t)}{t}dt \le c_0a(s)$, then there exists $\eta>0$ such that

$$\int_0^s \frac{a(t)}{t^{1+\eta}}dt \le c\frac{a(s)}{s^{\eta}},$$

and hence Theorem 5 about extrapolation applies.

Proof. By hypothesis

$$L \equiv \int_0^{s_1}\frac{1}{s}\int_0^s \frac{a(t)}{t}dtds \le c_0\int_0^{s_1} \le c_0^2a(s_1).$$

Also

$$L=\int_0^{s_1}\int_t^{s_1}\frac{a(t)}{ts}dsdt=\int_0^{s_1}\frac{a(t)}{t}\log\frac{s_1}{t}dt \le c_0^2a(s_1).$$

We repeat this argument and finally get

$$\int_0^s \frac{a(t)}{t}\frac{1}{j!}\log^j\frac{s}{t}dt \le c_0^{j+1}a(s).$$

Let $c_1>c_0$. Then

$$\int_0^s \frac{a(t)}{t}\sum\frac{1}{j!}\frac{1}{c_1^j}\log^j\frac{s}{t}dt \le ca(s),$$

and the sum $=(s/t)^{\eta}$ with $\eta=1/c_1$.